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THE PROBLEM OF THE INTERNAL CONSTITUTION OF STARS *

BY

N. R. SEN

It is a long time since stars were regarded as mere points of light in the distant sky, which served only to fix the bearings of the members of our solar system. Bessel, Struve and Henderson by their measurements of the parallax of three stars first confirmed the conjectures of some of the previous astronomers that the stars were at very great distance from the solar system. The parallax measurements first introduced a new scale of distance in Astronomy, and it became clear that the previous scale with the distance of the earth from the sun as astronomical unit was entirely inadequate for measurement of distances in the stellar world. The light year thus came into existence. The first impression about the stellar world thus obtained was that the stars are very much isolated and lonely objects in the sky and their mutual distances can be measured reasonably only in terms of light years. The vastness of this scale may be appreciated if we remember that the farthest extremities of the solar system yet known are only a few light hours from the sun. The failure of telescopes to magnify stellar objects could easily be understood.

The use of this parallax method is limited to a distance of only about a hundred light years. Further more powerful methods, as well as extension of our knowledge of the stellar world in general became possible only when celestial photography, photometry, and spectroscopic work came to be recognised as essential in astronomical observations. For instance distances for which the parallax method was useless could be measured indirectly by measurements of the absolute brightnesses of stars. In stars of the Cepheid type whose brightnesses go through regular periodic cycles there exists an intimate relation between their absolute brightnesses and periods, the absolute brightness in general increasing with the period. The determination of absolute brightness of these stars is now a matter of observation of their periods. Again, from the observation of the darkness of specific lines in the spectra of some stars the absolute brightness of these stars can be determined. The difference in the absolute brightnesses of two stars measures their difference in luminosities on a logarithmic scale, luminosity being defined as the total amount of radiation emitted by the star in one second and measured by such energy measuring devices as bolometers. Photometric measurements give on the other hand the apparent brightness of a star. From absolute and apparent brightnesses, the distance of a star can be deduced by simple calculation. The last named method of calculating distance by the comparison of intensities of specific lines in the spectra of stars is known as spectroscopic parallax.

* Presidential address delivered at the annual general meeting of the Calcutta Mathematical Society, held on January 29, 1946.

The spectral type of a star gives its surface temperature. There is a complicated method of calculating the diameter of a star from its surface temperature, apparent brightness, and distance. This calculation is based on our knowledge of the distribution of energy in the spectrum of a black body as expressed by Planck's law The application of this law is delicate as most stars only approximately represent a black body radiator. The accurate measurement of surface temperature is a difficult thing and the experimental skill necessary for this is enormous. However, in addition to this theoretical determination of the diameters of stars there is the wonderful direct method of measurement of diameters by Michelson's interferometer. Large number of measurements have not really been made as yet by this method. The measurements of stellar diameters reveal the fact that there is great variety among stars as regards their size. Stellar diameters, several hundred times the diameter of the sun are known without their showing corresponding variation in mass.

The giant stars which have these large diameters may differ from the dwarfs with smaller diameters in two ways. Either they contain intrinsically very much more matter than the dwarfs in which case their masses would also differ approximately in the same proportion, or the giants are only bloated configurations containing matter of extreme tenuity, and their masses are really comparable to those of the dwarfs; their huge bodies compensate for the large diminution in the density of their material. This question can be settled by determining the masses of the giant stars. To measure the mass we should have a gravitational field in which the mass can fall. Mass determinations of stars by observation have thus far been possible only in the case of binary stars, *i.e.*, a couple of neighbouring stars each moving in the gravitational field of the other. For visual binaries the relative orbit of the stars can be observed, or possibly the orbit of one star against the background of the sky. Their number is not large. Of great importance to astronomers are the spectroscopic binary stars which are at the same time eclipsing variables. For spectroscopic binaries, from the shift of the spectral lines can be determined the approaching or receding velocities of the stars in their orbits, while from the periods the size of their orbits (in a certain scale) in projection can be calculated. For the actual size of the orbits in space it is necessary to know the inclinations of the orbits to our line of sight. Exactly this is furnished by the light fluctuations in the case of eclipsing variables. An eclipsing variable is a double star whose brightness undergoes periodic change produced by the motion of the smaller component round the bigger when due to their relative position light is sometimes strengthened and sometimes weakened. The actual dimensions of the orbits are then obtainable for a spectroscopic binary which is at the same time an eclipsing variable. When the orbits and the periods are known, the masses are computed by the application of Kepler's law. It is also possible to find the absolute values of the diameters of the two stars when the dimension of the orbit has been found. From the determination of the masses of many giant stars Russel's conjecture that the giants are only bloated configurations with meterial contents comparable to those of the dwarfs was confirmed. Eddington made similar predictions regarding the dimension of the giant star Betelgeuse

from purely theoretical considerations, which was brilliantly verified by the measurements of Michelson and Pease. This prediction was the result of theoretical investigations by him which culminated in the discovery of his famous Mass Luminosity law, on the basis of which masses of stars can be conjectured from their luminosities. This law indeed gives very good approximations.

The problem for theoretical investigation of the stellar problem may be formulated thus. Given a star whose mass M; luminosity L, and radius R, have been determined, say from observations whose methods have just been described, we may ask what should be the appropriate density and temperature distributions in the star in order that a mass of matter M may be rolled up into a ball of radius R; from the surface of which will come out a total radiation of amount L per second. But this formulation is as yet not complete. There is still too much arbitrariness in the problem. We should be given the chemical composition of the star on the one hand, and the rate at which energy is produced inside on the other. The chemical composition which primarily is a constituent factor in the determination of gas pressure ultimately influences other characteristics of the star in a very important manner. The generation of energy within the star should on the one hand be consistent with the distribution of density and temperature inside which should permit this generation, and on the other hand be just sufficient to account for the radiation that comes out at the surface.

For these investigations the stars may be regarded to be in steady state. A variation may be considered only when evolutionary processes are involved. We may regard the matter in the interior of an ordinary star to be in an extremely hot gaseous condition. An important point to decide is if this mass should behave as perfect gas. The physical theory of the state of matter makes it almost certain that except under fairly well defined conditions, the exceptions are not however altogether rare, the perfect gas law should be quite appropriate as the equation of state of steller material for temperatures and densities inside the ordinary stars. We shall here limit ourselves exclusively to this case. The hydrostatic equilibrium and the equation of state for these stars will be given by the equation

$$\frac{dP}{dr} = -\frac{4\pi G}{r^2}\varrho\int r^2\varrho dr \tag{1}$$

which is obtained from the following hydrostatic equation of equilibrium and the mass equation

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\varrho, \tag{1a}$$

$$\frac{dM(r)}{dr} = 4\pi r^2\varrho, \tag{1b}$$

and

$$P = p_g + p_r, \quad p_g = \frac{k}{\mu H}\varrho T, \quad p_r = \tfrac{1}{3}aT^4. \tag{2}$$

The pressure is taken to be the sum of the gas pressure p_g, and the pressure due to

radiation p_r, which may be important for temperatures of the order of many million degrees inside the stars.

We require an equation which will say how the energy will flow through the star. As Eddington has shown the transmission of energy in the stars of this type by conduction will be small compared to that by radiation, and this latter method of transport in fact forms the chief mode of energy propagation. But conditions may arise whereby the temperature gradient for the transfer of energy by radiation would be steeper than the adiabatic temperature gadient under those conditions. The radiative transfer would then become unstable, and convective currents would be set up effecting most part of the transfer of energy. For *radiative transfer* the law is

$$\frac{dp_r}{dr} = -\frac{\kappa\varrho}{c}\frac{L(r)}{4\pi r^2} \tag{3a}$$

where p_r is the radiation pressure, ϱ the density, κ the opacity of the stellar material, and $L(r)$ the total rate of flow of radiant energy across the sphere of radius r. Where the radiative transfer is replaced by transfer by convection, a state of convective equilibrium will be established, which will be governed by the equation

$$\frac{dP}{P} = \Gamma_1\frac{d\varrho}{\varrho} \tag{3b}$$

with

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)}$$

where β is the ratio of the gas pressure to total pressure, and γ the ratio of the specific heats of the material. The opacity factor κ in (3a) is a function of the density, temperature and chemical composition of the stellar material, and may arise from two causes. Opacity may be caused by the photoelectric absorption of radiation by the stellar matter, or as in case of very high temperatures by scattering by electrons. The opacity function in (3a) is a sort of a mean called Rosseland mean, and the resultant opacity can be put, as was shown by Strömgren as

$$\kappa = \kappa_a + 1{\cdot}5\kappa_s \tag{3.1}$$

provided $\kappa_a > \kappa_s$, where κ_a is the part of opacity due to absorption and κ_s that due to scattering, but when $\kappa_s > \kappa_a$, the right hand side of (3.1) is to be replaced by $1{\cdot}5\kappa_a + \kappa_s$. The opacity due to absorption is a function of the pressure, temperature, and chemical composition of stars and is given by the physical theory as

$$\left.\begin{aligned} \kappa_a &= \frac{\kappa_a'}{\tau}\frac{\varrho}{T^{3\cdot5}}, \\ \kappa_a' &= 7{\cdot}65\times10^{25}(1-X-Y)[X+\tfrac{1}{2}Y+\eta_e(1-X-Y)] \end{aligned}\right\} \tag{3.2}$$

where τ called "guillotine factor" by Eddington is a very slowly varying function of temperature and pressure whose numerical value has been given by Strömgren and Morse, X, and Y the proportions of hydrogen and helium per unit mass of the stellar material. It is important to note that X and Y appear in the formula only indirectly.

The absorption frequencies of these elements lie in a region the effect of which is to contribute nothing to the *mean value of opacity*, only which appears in formula (3a). The contribution to *mean opacity* comes through the heavier elements whose proportion is represented by $(1-X-Y)$. The last factor in (3.2) which contains X and Y separately represents the *number of free electrons* per unit volume of the stellar material. The degree of ionisation of this material is represented by the constant η_e, which stands for the number of free electrons per protonic weight. The absorption due to scattering κ_s is independent of temperature and pressure (though it becomes significant only for high temperatures, say for above 25 million degrees), but dependent on the composition, and is given by

$$\kappa_s = 0{\cdot}385 \times [X + \tfrac{1}{2}Y + \eta_e(1-X-Y)]. \tag{3.3}$$

We thus see that the chemical composition of the star enters our calculation through the opacity factor in equation (3a). It also makes its appearance directly in a different way. We have assumed that the material of the star behaves as a perfect gas, and in this case the transfer of energy is by radiation or convection expressed by equation (3a) or (3b). The equation of state for the perfect gas, is given in (2), and the molecular weight μ in this formula is expressible as

$$\mu = \frac{1}{2X + \frac{3}{4}Y + \eta_R(1-X-Y)} \tag{2.1}$$

where η_R is the number of free particles (free electrons + nuclei) per protonic weight.

In equation (3a) occurs the function $L(r)$, called luminosity function meaning the total energy of radiation moving outwards per second across the sphere of radius r. This apparently will depend upon the rate at which energy is being generated inside this sphere. We thus need to know about the rate of generation of energy. If $\varepsilon(\varrho, T)$ be this generation function giving the energy generated per unit mass per unit time we must have

$$\frac{dL(r)}{dr} = 4\pi r^2 \varrho \varepsilon(\varrho, T). \tag{4}$$

The function $\varepsilon(\varrho, T)$ should be known from physical theories. In recent years a formula for this has been worked out by Bethe, based on the previous works of Gamow, Atkinson and others on the transmutation of the nuclei of Hydrogen, the lightest element known, to Helium through the catalytic agents carbon and nitrogen which remain unaltered at the end of the process. Four hydrogen nuclei ultimately produce one helium nucleus and set free energy as radiation. Bethe's formula is

$$\varepsilon(\varrho, T) = \varepsilon_0 X \varrho T^{-2/3} e^{-B/T^{\frac{1}{3}}} \tag{4.1}$$

where the numerical values of ε_0 and B have been given by Bethe.

Equations (1a), (1b), (2), (3a), (4), and (4.1) really place four differential equations at our disposal from which we can determine the density, and temperature distribution inside, when there is no convection. In the case of convection we take (1a), (1b), (2),

(8b), (4) and (4.1). Now the equations break up into two groups, the first four determining the temperature and density fields, and the last two the luminosity.

The boundary conditions to be satisfied for the solutions of our equations are $M = 0$, $L = 0$, at $r = 0$, and $\varrho \to 0$, $T \to 0$ at the outer boundary $r = R$. Inspite of the temperature of a few thousand degrees at the surface which is small compared to millions of degrees in the interior we can without serious error put $T = 0$ at $R = r$ as the outer boundary condition.

If we replace ε in (4) by (4·1) and P by ϱ and T in equation (1a), then with these and (1b) and (8a) we have four differential equations involving ϱ, T, $L(r)$, $M(r)$, X and Y. We may consider X, and Y to be known constants, working with the average values of these quantities in the configuration. The equations will have fou parametric solutions. We may call these parameters C_1, C_2, C_3, and C_4. The conditions at the centre $L(0) = M(0) = 0$, reduce them to two. Further, the external boundary conditions at $r = R$ for ϱ and T will determine these two parameters in terms of the radius R (if for the time being we assume that this is possible for finite R). Thus each solution will contain *three* disposable constants, X, Y, *i.e.*, the hydrogen and helium contents of the stellar material, and R, the radius. This is of course true of any other energy generation law of type (4.1).

A very important configuration from the point of view of stellar structure is the one, first given by Cowling, in which there is a small central core wherein convective currents are responsible for the main transport of energy, while outside this core the transfer of energy is by radiation. This composite configuration also fits in excellently with the scheme formulated for the mode of energy generation inside stars by Bethe. The generation of energy may take place inside this convective core which will constantly be stirred by the convective currents ensuring well mixing of the material and unhampered continuation of energy generation. The solution of the equations corresponding to such a model does not also give us ultimately more than three parameters. The solution for the purely convective core worked out with equations (1a, 1b), (2) and (8b), three differential equations of the first order, will contain as before three arbitrary constants which will be reduced to two on account of the central condition $M(0) = 0$. We call these constants C_1' and C_2'. The outer solution for radiative envelope as in the case above will contain only three constants C_1, C_2 and C_3, and a parameter L, the constant luminosity in the envelope, the temperature gradient in which will be given by equation (4) with $L(r) = L$. We need not now consider equation (4.1) as there is no generation of energy in the envelope. The outer boundary conditions and the given total mass M of the configuration will completely determine the three constants, so that the exterior solution will be expressible in terms of L, M and R only. But certain conditions of continuity of the pressure, temperature, mass and luminosity have to be fulfilled at the interface where the convective and the radiative gradients merge into one another. The conditions of continuity at the interface supply *four* equations which we may call equations of fit (in the literature on the subject *two* equations derived from these four by elimination are known as equations of fit). These four equations of fit

will determine the two constants C_1' and C_2', and in addition provide two relations among the parameters L, M, R, X and Y (X and Y being regarded as constants in the equations). We thus see that whether the configuration be of the purely radiative, or of the composite convective-radiative type, the solutions of our equations and the proper boundary conditions finally supply us with only *three* parameters in terms of which every other property of the configuration is expressible. This is only an elaboration of the well known Vogt-Russell Theorem.

Now we come to the essential point of our problem. We have seen observation supplies us with three constants of a star, *viz.*, luminosity (L), mass (M), and radius (R). The theory on the other hand gives us two relations among the five quantities L, M, R, X and Y. Hence the observed values of L, M, and R will uniquely fix X and Y. We thus see that the equations of stellar equilibrium given before will for assigned set of values for L, M and R of a star, fix X and Y, that is its chemical composition as regards hydrogen and helium content. The temperature and density fields inside the star are of course given by the solution at the same time. In short in the present day theory of the internal constitution of stars there appear just as many unknown parameters as will be completely fixed by the observational material. We may regard L, M, R, X and Y, as *five* parameters of a star, *any two* of which will be fixed by the other three. Naturally, we take L, M and R to be the parameters which fix the composition X and Y.

We thus see that the stellar problem as presented above is not overdetermined. This has one disadvantage. If we desire to have an internal check in our theory to be furnished by the three observational prameters L, M and R, we find we have no chance for that here. The relation among the parameters of the solution furnished by the theory only helps to determine X and Y, two unknown factors of the composition. For the desired check we must look to other quarters. Two different considerations have indeed been proposed by Strömgren and Biermann for check. Strömgren on the basis of a different theory of energy generation due to Weizsäcker attempted to determine the limits of the ratio of X to Y, and of Y to heavier elements in the interior of stars from the reasonable assumption that these ratios cannot exceed the corresponding values obtaining in the composition of the stellar atmosphere. The limiting value of Y he obtained from this is quite high. Biermann, on the other hand, obtains from the condition of dynamical stability of a star worked out in terms of mean ionisation energy and mean temperature within it, a lower limit for its hydrogen content [*Zeit. f. Astrophys.*, **16**, 29, (1938); a second paper by Biermann which appeared in the same journal in 1948 has not been available to me in India]. These are checks limiting the values of the parameters X and Y. A complete quantitative internal check in confirmation of the theory of stellar constitution described above is not known.

Though it is not the object of this address to discuss the various types of solutions of the equations of stellar equilibrium certain first approximation solutions which have proved to be quite satisfactory in many cases, may be mentioned here. Eddington

proposed a model known as "standard model," in which equations (4), (4a) regarding generation of energy are dispensed with. In the absence of the knowledge of energy generation, Eddington made a conjecture that

$$\kappa\eta = \kappa \frac{L(r)/M(r)}{L/M}$$

should be extremely slowly variable within a star, so that $\kappa\eta$ may be put equal to a constant without committing a very serious error. This gave a fresh relation for the function $L(r)$, and so it was possible to dispense with (4) and (4·1) and solve the equations (1), (2) and (3a) completely. The configuration came out to be a polytrope $n = 3$, the mass and the composition in this case entirely determining the ratio of radiation to gas pressure. The net outcome of this was the famous Mass-luminosity relation of Eddington according to which the mass of a star can be predicted from its luminosity. This relation which is true for more general model for negligible radiation pressure, as was shown by Strömgren, has also the form

$$L/L_{\odot} = 1{\cdot}7\times 10^{25}.\ \frac{1}{\kappa_0\eta_c}(M/\odot)^{5\cdot5}(R_{\odot}/R)^{0\cdot5}(\mu\beta)^{7\cdot5} \tag{5}$$

where $\kappa_0 = (\kappa_a'/\tau)$ in equation (3·2), η_c the value of η at the centre, and $\odot$ the mass of the sun (the suffix $\odot$ referring to the corresponding solar value). This formula has also been used for determining the hydrogen content of stars.

A second model also worked out by Eddington is one in which the luminosity function L is considered constant throughout the star, close upto the centre. The integration of this point source model clearly indicates that near the centre the radiative gradient must become unstable, so that no stable configuration can be constructed with radiative gradient reaching upto the centre. The mathematical solution with radiative gradient right up to the centre for constant opacity κ has the characteristic (for $M = 0$, at $r = 0$) $\varrho \to 0$ as $r \to 0$. The most important model which fits in with the scheme discussed before is the one worked out by Cowling in which, as stated before, a convective core is surrounded by an atmosphere in which the radiative gradient prevails, the radiation pressure in the whole model being negligible. As a convective configuration with negligible radiation pressure is a polytrope $n = \frac{3}{2}$, this amounts to fitting a solution of (1), (2) and (3a) with constant L from outside to a polytrope $n = \frac{3}{2}$ occupying the central region. The equations admit of homologous transformation so that it becomes possible to find the solution which fixes the relative scale of the core and the outside. It is found that the convective core contains about 14·5 percent of the total mass and has an extension given by about 19 percent of the radius of the configuration.

But Cowling's composite stellar model does not involve any consideration of energy. The total rate of generation of energy of the Cowling model may be calculated from equations (4) and (4.1), and if the model be consistent with Bethe's energy generation formula the value of L so obtained should agree with the constant value assumed in equation (3a). But a complete formulation of the problem should as stated above require an outside solution of (1), (2) and (3a) with an appropriate constant value of L

to be fitted to a solution of (1), (2) and (3b) going right upto the centre, such that the integration of (4) and (4a) should give the same value of L. This latter solution for the core in the case of negligible radiative pressure should be that for a polytrope $n = \frac{3}{2}$. Only such a solution will be entirely consistent with Bethe's energy generation formula. We may make a scheme in which for assigned composition and central temperature integrations may be started from the centre. This assignment should fix the configuration completely. When the central temperature is not too high the core will be a polytrope $n = \frac{3}{2}$, for which $L(r)$ may be calculated by (4) and (4·1). The success of the solution will depend on the choice of the right central density which will lead to the correct boundary condition $\varrho \to 0$, $T \to 0$ simultaneously outside. A series of such solutions will supply us with data for comparison with known stars. An approximate solar model in complete agreement with Bethe's energy generation formula has been worked out with hydrogen-content $X = 0{\cdot}35$, $Y = 0$. It is found that for a star of this composition, and central temperature $T_c = 20{\cdot}2 \times 10^6$, Bethe's energy generation formula will be valid when its mass, luminosity and radius are given by

$$M = 2{\cdot}12 \times 10^{33} \text{gms}, \quad L = 3{\cdot}7 \times 10^{33} \text{ erg/sec}, \quad R = 8 \times 10^{10} \text{ cm}.$$

The central density then is $\varrho_c = 45{\cdot}5 \text{ gm/cm}^3$. This may be regarded as an approximate model for the sun. It may further be shown that for stars of small masses for which radiation pressure is negligible, the Cowling model is quite in good agreement with Bethe's energy generation scheme. This model may be utilised in constructing solutions for many dwarf stars with appropriate hydrogen content and zero helium content. The mass and luminosity of such stars do not differ much from those of the sun; the hydrogen content is very near 35 percent in all of them, though their density and temperature fields are different from those of the solar model. The convective-radiative models so far constructed for larger stellar masses in which radiation pressure should be appreciable are not in agreement with Bethe's energy generation formula (4·1).

This briefly represents the present state of the theory of the internal constitution of stars, rather of dwarfs. We have only stated the problem and briefly touched upon certain special solutions. A theory of stellar structure cannot, of course, be complete until it explains the peculiar pattern of stellar representation on the Hertzsprung-Russel diagram. Strömgren has explained the groupings on the diagram in terms of two parameters, mass, and composition represent by the average molucular weight. A complete knowledge of energy generation in stars may be expected to give us finer points of the Hertzsprung-Russel diagram, and a clue to the life history of a star.

A NOTE ON AVERAGE STRESSES IN A PLATE

BY

S. GHOSH

(*Received January 21, 1946*)

INTRODUCTION

The object of the present note is to make a critical examination of the assumptions made in solving the problem of a plate stretched and bent by forces acting on its cylindrical edge. It is seen that, in the flexure of a plate, a state of generalized plane stress is always possible, while only a restricted distribution of deforming forces and couples can lead to a state of plane stress. It is also pointed out that, in the stretching of a plate by forces in its plane, a state of plane stress can always be found, while the existence of a state of generalized plane stress has not been proved. It is further pointed out that the equations for the determination of average stresses $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$ and average displacements $\bar{u}$, $\bar{v}$ involve $\overline{Z}_z$, so that unless some suitable assumption is made about $\overline{Z}_z$, the problem of average stresses cannot be solved. Besides the assumptions in plane stress and generalized plane stress, other simplifying assumptions are mentioned, and it is shown that the most general value of $\overline{Z}_z$ is a plane harmonic function or the average value of the product of a harmonic function by x or y or z or r^2.

THE PROBLEM

Let the middle plane of the plate be taken as the plane $z = 0$, and let its plane faces be given by $z = \pm h$. Let the plane faces be free from tractions which are assumed to be applied only on the cylindrical edge.

The stress equations of equilibrium are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z} = 0, \tag{1a}$$

$$\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z} = 0, \tag{1b}$$

$$\frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z} = 0, \tag{1c}$$

and the boundary conditions are

$$X_z = Y_z = Z_z = 0, \tag{2}$$

on the faces $z = \pm h$, and

$$lX_x+mX_y = X_\nu, \quad lX_y+mY_y = Y_\nu, \quad lX_z+mY_z = Z_\nu \tag{3}$$

on the cylindrical edge, where l, m, n are the direction cosines of the normal to it.

In order that a solution of the stress equations (1), subject to the boundary conditions (2) and (3), may give a physically possible stress distribution within the plate, the components of strain calculated from the relations

$$Ee_{xx} = X_x - \sigma Y_y - \sigma Z_z, \qquad Ee_{yz} = 2(1+\sigma)Y_z, \tag{4a}$$

$$Ee_{yy} = Y_y - \sigma Z_z - \sigma X_x, \qquad Ee_{zx} = 2(1+\sigma)Z_x, \tag{4b}$$

$$Ee_{zz} = Z_z - \sigma X_x - \sigma Y_y, \qquad Ee_{xy} = 2(1+\sigma)X_y, \tag{4c}$$

must satisfy the compatibility equations

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{yz}}{\partial y \partial z}, \qquad \frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{1}{2}\frac{\partial}{\partial x}\left[-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right], \tag{5a}$$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = \frac{\partial^2 e_{zx}}{\partial z \partial x}, \qquad \frac{\partial^2 e_{yy}}{\partial z \partial x} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right], \tag{5b}$$

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}, \qquad \frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{xy}}{\partial z}\right]. \tag{5c}$$

The compatibility conditions for the strain components may be replaced by the compatibility conditions for stresses. For an isotropic body these are

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x^2} = 0, \qquad \nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y \partial z} = 0, \tag{6a}$$

$$\nabla^2 Y_y + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y^2} = 0, \qquad \nabla^2 Z_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial z \partial x} = 0, \tag{6b}$$

$$\nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial z^2} = 0, \qquad \nabla^2 X_y + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x \partial y} = 0, \tag{6c}$$

where

$$\Theta = X_x + Y_y + Z_z. \tag{7}$$

As it is often found to be very difficult to obtain a solution of the equations (1), subject to the boundary conditions (2) and (3) and the compatibility conditions (6), we have to make some simplifying assumptions for finding a solution of the problem. When the boundary conditions are prescribed, the solution of the problem is unique, and we are not permitted to make any such assumptions. But, if the thickness of the plate is small and the variations of the stresses and displacements along the thickness of the plate are relatively unimportant, we can replace, in the boundary conditions (3), X_v, Y_v, Z_v along a generator of the cylindrical edge by any suitable set of statically equivalent tractions, which will help us in solving the equations (1) and (6). Then, by appealing to St-Venant's principle, we can take the solution of the problem to be practically the same as that obtained under the simplified assumptions, at points of the plate not close to the edge. Again, it is difficult to determine what relaxations are to be permitted in the distribution of tractions along a generator of the cylindrical edge, so as to lead to an easy solution of the equations (1) and (6), and to leave the resultant

traction along a generator unaltered. We are therefore forced to make some assumption, which will simplify the equations (1) and (6) in such a manner that their solution can be easily obtained. Moreover, the solution so found must be such that the tractions along a generator of the cylindrical edge, calculated from equations (8), must be statically equivalent to the given tractions along the same generator.

PLANE STRESS

When the thickness of the plate is very small, X_z, Y_z, Z_z are, in general, very small throughout the plate, A plausible assumption about the distribution of stress within the plate is that

$$X_z = Y_z = Z_z = 0, \tag{8}$$

identically. Then we can show (Love, 1927, pp. 206-7) that the stress equations (1) and the compatibility conditions (6) are satisfied, if we take

$$X_x = \frac{\partial^2 \chi}{\partial y^2}, \quad Y_y = \frac{\partial^2 \chi}{\partial x^2}, \quad X_y = -\frac{\partial^2 \chi}{\partial x \partial y}, \tag{9}$$

where

$$\chi = \chi_0 + z\chi_1 - \frac{1}{2}\frac{\sigma}{1+\sigma} z^2 \Theta_0. \tag{10}$$

The functions χ_0, χ_1 are independent of z, and satisfy the equations

$$\nabla_1^2 \chi_0 = \Theta, \quad \nabla_1^2 \chi_1 = \beta, \tag{11}$$

where Θ_0 is a plane harmonic function in x, y, and β is a constant.

The assumptions (8) satisfy the boundary conditions (2). The stress-resultants and the stress-couples within the plate are given by (Love, 1927, p. 468 and p. 470)

$$T_1 = \frac{\partial^2 \chi'}{\partial y^2}, \quad T_2 = \frac{\partial^2 \chi'}{\partial x^2}; \quad S_1 = -\frac{\partial^2 \chi'}{\partial x \partial y}, \quad N_1 = N_2 = 0, \tag{12a}$$

and

$$G_1 = \tfrac{2}{3} h^3 \frac{\partial^2 \chi_1}{\partial y^2}, \quad G_2 = \tfrac{2}{3} h^3 \frac{\partial^2 \chi_1}{\partial x^2}, \quad H_1 = \tfrac{2}{3} h^3 \frac{\partial^2 \chi_1}{\partial x \partial y}, \tag{12b}$$

where

$$\chi' = 2h\chi_0 - \frac{1}{3}\frac{\sigma}{1+\sigma} h^3 \Theta_0. \tag{12c}$$

The stress-resultants and stress-couples per unit length of the edge-line (*i.e.*, the curve in which the edge cuts the middle plane of the plate) are given by

$$T = l^2 \frac{\partial^2 \chi'}{\partial y^2} + m^2 \frac{\partial^2 \chi'}{\partial x^2} - 2lm \frac{\partial^2 \chi'}{\partial x \partial y}, \tag{13a}$$

$$S = lm\left(\frac{\partial^2 \chi'}{\partial x^2} - \frac{\partial^2 \chi'}{\partial y^2}\right) - (l^2 - m^2)\frac{\partial^2 \chi'}{\partial x \partial y}; \tag{13b}$$

$$N = 0, \tag{13c}$$

$$G = \tfrac{2}{3}h^3\left\{l^2\frac{\partial^2\chi_1}{\partial y^2}+m^2\frac{\partial^2\chi_1}{\partial x^2}-2lm\frac{\partial^2\chi_1}{\partial x\partial y}\right\}, \tag{13d}$$

$$H = \tfrac{2}{3}h^3\left\{lm\left(\frac{\partial^2\chi_1}{\partial y^2}-\frac{\partial^2\chi_1}{\partial x^2}\right)+(l^2-m^2)\frac{\partial^2\chi_1}{\partial x\partial y}\right\}. \tag{13e}$$

The equation satisfied by χ_0 is sufficiently general to lead to arbitrarily prescribed values of T, S on the edge line, but the equation satisfied by χ_1 is such that it gives only restricted values of G and $\partial H/\partial s$ on the edge-line. If we assume that the tractions applied to the edge of the plate are distributed according to the equations (8), (9), (10) and (11), we obtain an exact solution of the problem. For any other distribution of tractions on the edge, statically equivalent to the stress-resultants T, S, N and the stress couples G, H per unit length of the edge-line, as given by equations (13), the solution represents the state of stress in the plate with sufficient approximation at points not near the edge.

Thus the state of plane stress corresponds to the exact solution of the stress equations of equilibrium (1) and the compatibility conditions (6), subject to the boundary conditions (2) and the condition that the resultant tractions on the edge per unit length of the edge-line reduce to an arbitrary force in the plane of the plate and a restricted couple given by (13d) and (13e). The distribution of tractions on the cylindrical boundary, given by (8), (9), (10) and (11) is perfectly general as regards x, y are concerned, but is restricted to be a quadratic in z.

Generalized plane stress

Filon (1903) suggests that, if we are content with the average values of stresses in the plate, we can relax, to some extent, the restrictions placed on the stress distribution in the theory of plane stress. He replaces (8) by the condition that

$$Z_z = 0 \tag{14}$$

identically, while X_z, Y_z satisfy the conditions (2).

Integrating the stress equations of equilibrium (1) with respect to z from $-h$ to h, and observing that $X_z = Y_z = Z_z = 0$ on $z = \pm h$, we get

$$\frac{\partial\overline{X}_x}{\partial x}+\frac{\partial\overline{X}_y}{\partial y}=0,\quad \frac{\partial\overline{X}_y}{\partial x}+\frac{\partial\overline{Y}_y}{\partial y}=0,\quad \frac{\partial\overline{X}_z}{\partial x}+\frac{\partial\overline{Y}_z}{\partial y}=0, \tag{15}$$

where

$$\overline{X}_x = \frac{1}{2h}\int_{-h}^{h}X_x\,dz,$$

with similar expressions for the others.

Since $Z_z = 0$, we can write (Coker and Filon, 1931, p 127)

$$\overline{X}_x = \lambda'\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}\right)+2\mu\frac{\partial\bar{u}}{\partial x}, \tag{16a}$$

$$\overline{Y}_y = \lambda'\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}\right)+2\mu\frac{\partial\overline{v}}{\partial y}, \tag{16b}$$

$$\overline{X}_y = \mu\left(\frac{\partial\overline{v}}{\partial x}+\frac{\partial\overline{u}}{\partial y}\right), \tag{16c}$$

where $\lambda' = 2\lambda\mu/(\lambda+2\mu)$.

The first two equations of (15) and the equations (16) are of the same form as those in a plane strain problem, with this difference that λ is replaced by λ'. Then we can write

$$\overline{X}_x = \frac{\partial^2\overline{\chi}}{\partial y^2}, \quad \overline{Y}_y = \frac{\partial^2\overline{\chi}}{\partial x^2}, \quad \overline{X}_y = -\frac{\partial^2\overline{\chi}}{\partial x\partial y}, \tag{17}$$

where

$$\nabla_1^4\overline{\chi} = 0. \tag{18}$$

If we consider flexural actions, we have to introduce, in addition to the equations (15), the equations satisfied by flexural couples. We replace the equations (15) by the equations of equilibrium of a plate in terms of stress-resultants and stress-couples, which are (Love, 1927, p. 458)

$$\frac{\partial T_1}{\partial x}+\frac{\partial S_1}{y} = 0, \quad \frac{\partial S_1}{\partial x}+\frac{\partial T_2}{\partial y} = 0, \quad \frac{\partial N_1}{\partial x}+\frac{N_2}{\partial y} = 0, \tag{19}$$

and

$$\frac{\partial H_1}{\partial x}-\frac{\partial G_2}{\partial y}+N_2 = 0, \quad \frac{\partial G_1}{\partial x}-\frac{\partial H_1}{\partial y}-N_1 = 0, \tag{20}$$

where

$$T_1 = \int_{-h}^{h} X_x dz = 2h\overline{X}_x, \quad T_2 = \int_{-h}^{h} Y_y dz = 2h\overline{Y}_y, \quad N_1 = \int_{-h}^{h} X_z dz = 2h\overline{X}_z,$$

$$N_2 = \int_{-h}^{h} Y_z dz = 2h\overline{Y}_z, \quad S_1 = \int_{-h}^{h} X_y dz = 2h\overline{X}_y, \quad H_1 = -\int_{-h}^{h} zX_y dz,$$

$$G_1 = \int_{-h}^{h} zX_x dz, \quad G_2 = \int_{-h}^{h} zY_y dz.$$

The equations (19) and (20) divide themselves into two groups, one containing T_1, T_2, S_1 and the other, N_1, N_2, G_1, G_2, H_1. The first group corresponds to the stretching of the plate by forces on its rim, such that the resultant traction on a generator of the rim is a single force in the middle plane of the plate. The second group corresponds to the flexure of the plate by forces on its rim, such that tractions on a generator of the rim reduce to a force acting at a point of the edge-line normal to the middle plane of the plate and a couple.

Filon restricts himself to the stretching of the plate by forces in its plane. He claims that the generalized plane stress gives a more general distribution of stress within the plate than is given by plane stress. The stress-couples vanish in this case, and the stress-resultants per unit length of the edge-line are given by

$$T = 2h\left\{l^2\frac{\partial^2\bar{\chi}}{\partial y^2}+m^2\frac{\partial^2\bar{\chi}}{\partial x^2}-2lm\frac{\partial^2\bar{\chi}}{\partial x\partial y}\right\}, \tag{21a}$$

$$S = 2h\left\{lm\left(\frac{\partial^2\bar{\chi}}{\partial x^2}-\frac{\partial^2\bar{\chi}}{\partial y^2}\right)-(l^2-m^2)\frac{\partial^2\bar{\chi}}{\partial x\partial y}\right\}, \tag{21b}$$

$$N = 0, \tag{21c}$$

where $\bar{\chi}$ satisfies the equation (18).

These formulae are of sufficient generality to satisfy arbitrarily prescribed values of T, S on the edge-line of the plate. Filon claims that we get, in this way, the average stresses $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$ and the average displacements $\bar{u}$, $\bar{v}$ from the equations (16) and (17), with assumptions which are less stringent than those in the case of plane stress. We do not require that X_z, Y_z should vanish throughout the plate, as in the state of plane stress, but that they should vanish only on the faces $z = \pm h$.

If we restrict ourselves to flexural actions only, we can find (Love, 1927, p. 478) an exact solution of the equations (1), (6) and (14) which satisfies the boundary conditions (2) and gives $T_1 = T_2 = S_1 = 0$. The expressions for the stress-resultant N and the stress couples G, H are sufficiently general to admit of arbitrarily prescribed values of G, $N-(\partial H/\partial s)$ on the edge-line.

Comparison between the state of plane stress and a state of generalized plane stress

Let us first consider the stretching of the plate by forces in its plane. Let the stress-resultants T, S per unit length of the edge-line of the plate be prescribed arbitrarily, and let the stress-resultant N and the stress-couples G, H vanish on that line. Then the average stresses $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$ and the average displacements $\bar{u}$, $\bar{v}$ have the same values, both in the state of plane stress and in the state of generalized plane stress. But Filon claims that the distribution of stresses in the plate is more general in a state of generalized plane stress than in the state of plane stress. He tries to obtain a solution of the problem in which $Z_z = 0$ identically, and $X_z = Y_z = 0$ on the faces $z = \pm h$, but the solution he actually obtains gives $X_z = Y_z = 0$ identically. Coker and Filon (1931, p. 134) remark that "... this is only one of many solutions which make $Z_z = 0$ throughout. The others do not make X_z, Y_z vanish identically." Southwell (1936, pp. 201-15) has obtained the most general solution of the equations (1) and (14) which satisfies the compatibility conditions (6). In this paper, he remarks (p. 204) that "... it seems to me, the generality of those other solutions is the real question at issue." From his analysis, he concludes (pp. 205-6) that "... the less stringent assumptions of generalized plane stress give its solution for *flexural* actions ... the same degree of generality as is possessed, for actions which leave the middle surface plane, by the ordinary theory of plane stress. *No wider application is forthcoming in respect of actions of the latter (extensional) kind.*"

Let us examine the problem more carefully. When we determine the average stresses $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$, and the average displacements $\bar{u}$, $\bar{v}$ from the equations (16), (17) and (18) and the prescribed values of T, S as given by (21a) and (21b), we do not prove the existence of a solution of the problem (with which we start) in which Z_z is identically zero, while X_z, Y_z are not. What we really prove is that, *if such a solution corresponding to a state of generalized plane stress exists*, then this solution gives the same average values $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$, $\bar{u}$, $\bar{v}$ as those given by (16), (17), (18), (21a) and (21b). The equations (17) are obtained from the first two equations of (15) which, in their turn, are obtained from the first two equations of (1). The equation (18) is obtained from the equations (17), the stress-strain relations (4) and only *one* of the compatibility equations (5), *viz.*,

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y} . \tag{22}$$

Thus, it appears that the average stresses which we obtain in the problem of generalized plane stress are the average values of stresses satisfying the first two equations of (1), the equations (2), (4), (14) and (22), subject to the condition that T, S calculated from these stresses are prescribed on the edge-line Since we have not shown that the solution from which we calculate the averages, satisfies the third equation of (1) and the remaining five compatibility equations (5), we cannot conclude that such a system is physically possible in a plate. Only when $X_z = Y_z = 0$ identically in addition to $Z_z = 0$, we have proved that the relevant equations are all satisfied, and the average values $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$, $\bar{u}$, $\bar{v}$ which we get are the average values of X_x, Y_y, X_y, u, v in the state of plane stress of the plate. As Filon defines a state of generalized plane stress to be given by an exact solution of the equations (1), (6), (14) and the condition that X_z, Y_z are not identically zero, *we cannot assert that a state of generalized plane stress, distinct from the state of plane stress, is possible in the plate*. To be convinced of the existence of such a solution, we must obtain at least one solution in which X_z, Y_z are not identically zero.

Let us next suppose that T, S vanish on the edge-line and that N, G, H are arbitrarily prescribed on that line. Then we have seen that a state of plane stress is impossible in the plate, unless $N = 0$ and G, $\partial H/\partial s$ are suitably distributed along the edge-line (Love, 1927, p. 471). It has also been shown (Love, 1927, p. 478) that, in this case, an exact solution of the fundamental equations can be obtained, so that a generalized plane stress is always possible.

Thus, when the middle plane of the plate is bent, a state of generalized plane stress is always possible, while a state of plane stress is possible under exceptional circumstances; but when the plate is stretched by forces in its plane, a state of plane stress is always possible, while the existence of a state of generalized plane stress has not been proved.

Average stresses and displacements in a plate

Restricting ourselves to the stretching of the plate by forces in its plane, we have

$$\frac{\partial \overline{X}_x}{\partial x}+\frac{\partial \overline{X}_y}{\partial y}=0, \quad \frac{\partial \overline{X}_y}{\partial x}+\frac{\partial \overline{Y}_y}{\partial y}=0, \tag{23}$$

subject to the conditions

$$l\overline{X}_x+m\overline{X}_y=\overline{X}_v, \quad l\overline{X}_y+m\overline{Y}_y=\overline{Y}_v, \tag{24}$$

on the rim of the plate.

The average values of e_{xx}, e_{yy}, e_{xy} in terms of average stresses are determined from the equations (4) and are given by

$$E\bar{e}_{xx}=\overline{X}_x-\sigma\overline{Y}_y-\sigma\overline{Z}_z, \tag{25a}$$

$$E\bar{e}_{yy}=\overline{Y}_y-\sigma\overline{Z}_z-\sigma\overline{X}_x, \tag{25b}$$

$$E\bar{e}_{xy}=2(1+\sigma)\overline{X}_y. \tag{25c}$$

The only equation of (5) which, on averaging, does not involve the surface values of derivatives of strain components, is the equation (22). From this equation, we get

$$\frac{\partial^2 \bar{e}_{xx}}{\partial y^2}+\frac{\partial^2 \bar{e}_{yy}}{\partial x^2}=\frac{\partial^2 \bar{e}_{xy}}{\partial x \partial y}. \tag{26}$$

The equations (23) give

$$\overline{X}_x=\frac{\partial^2 \bar{\chi}}{\partial y^2}, \quad \overline{Y}_y=\frac{\partial^2 \bar{\chi}}{\partial x^2}, \quad \overline{X}_y=-\frac{\partial^2 \bar{\chi}}{\partial x \partial y}. \tag{27}$$

Calculating $\bar{e}_{xx}$, $\bar{e}_{yy}$, $\bar{e}_{xy}$ from the equations (25) and (27) and substituting in (26), we get

$$\nabla_1^4\bar{\chi}=\sigma\nabla_1^2\overline{Z}_z. \tag{28}$$

This equation has been obtained by Green (1945, p. 226).

To obtain the average displacements $\bar{u}$, $\bar{v}$, we have

$$E\frac{\partial \bar{u}}{\partial x}=-(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial x^2}+\nabla_1^2\bar{\chi}-\sigma\overline{Z}_z, \tag{29a}$$

$$E\frac{\partial \bar{v}}{\partial y}=-(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial y^2}+\nabla_1^2\bar{\chi}-\sigma\overline{Z}_z, \tag{29b}$$

$$E\left(\frac{\partial \bar{v}}{\partial x}+\frac{\partial \bar{u}}{\partial y}\right)=-2(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial x \partial y}. \tag{29c}$$

The equation (28) shows that

$$\xi=\nabla_1^2\bar{\chi}-\sigma\overline{Z}_z \tag{30}$$

is a plane harmonic function, so that we consider it to be the real part of an analytic function

$$\xi + i\eta = f(Z) \tag{31}$$

of the complex variable $Z = x+iy$. Let us consider the analytic function

$$F+iG = \int_{Z_0}^{Z} f(Z)dZ. \tag{32}$$

Then

$$\xi+i\eta = f(Z) = \frac{d}{dZ}(F+iG) = \frac{\partial F}{\partial x} + i\frac{\partial G}{\partial x}. \tag{33}$$

Since

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}, \quad \frac{\partial G}{\partial x} = -\frac{\partial F}{\partial y}, \tag{34}$$

we can write the equations (29) as

$$E\frac{\partial \bar{u}}{\partial x} = -(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial x^2} + \frac{\partial F}{\partial x},$$

$$E\frac{\partial \bar{v}}{\partial y} = -(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial y^2} + \frac{\partial G}{\partial y},$$

$$E\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) = -2(1+\sigma)\frac{\partial^2 \bar{\chi}}{\partial x \partial y} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial x},$$

the term $\partial F/\partial y + \partial G/\partial x$ which is added to the last equation being zero. Hence we get for the average displacements

$$E\bar{u} = -(1+\sigma)\frac{\partial \bar{\chi}}{\partial x} + F, \quad E\bar{v} = -(1+\sigma)\frac{\partial \bar{\chi}}{\partial y} + G. \tag{35}$$

Formulae for displacements, when $\bar{Z}_z$ is a plane harmonic function, have been given by Green (1945, p. 226).

Thus, unless we know the average value of Z_z, we cannot determine in this manner the average stresses $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$ and the average displacements $\bar{u}$, $\bar{v}$. Various assumptions can be made about $\bar{Z}_z$, and for each such assumption $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$, $\bar{u}$, $\bar{v}$ are uniquely determined. But it must be observed that it is very doubtful whether these average values are the average values of actual stresses and displacements in the plate, when it is stretched by forces in its plane.

(1) If we take $Z_z = 0$, then $\bar{Z}_z = 0$. This is the case of plane stress or generalized plane stress according as X_z, Y_z are identically zero or not. We have already seen that a state of plane stress is always possible, but the existence of a state of generalized plane stress appears probable, but remains to be proved.

(2) If we take $\bar{Z}_z = 0$, we get the same average stresses $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$ and the same average displacements $\bar{u}$, $\bar{v}$ as in the case of plane stress or generalized plane stress

(Trefftz, 1928, p. 112, and Southwell, 1936, p. 203, foot note). But the question remains unsettled whether such a state of stress is possible or not.

(3) If we take $\overline{Z}_z$ to be a plane harmonic function, we get the same average stresses $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$ as in cases (1) and (2), but different average displacements $\overline{u}$, $\overline{v}$. This has been pointed out by Green (1945, p. 226). The validity of such a state of stress remains to be examined.

(4) If we take any other *plausible* value of $\overline{Z}_z$, we get average stresses $\overline{X}_x$, $\overline{Y}_y$, $\overline{X}_y$ and average displacements $\overline{u}$, $\overline{v}$ different from those in cases (1), (2) and (3). But to determine a plausible value of $\overline{Z}_z$ which will lead to a physically possible state of stress in the plate is a very difficult question.

To determine the plausibility of a value of $\overline{Z}_z$, we observe that

$$\nabla^4 Z_z = 0. \tag{36}$$

From the theory of partial differential equations, we know that every solution of (36) can be expressed in any one of the forms

$$f_1 + xf_2, \quad f_1 + yf_2, \quad f_1 + zf_2, \quad f_1 + r^2 f_2, \quad (r^2 = x^2 + y^2 + z^2)$$

where f_1, f_2 are harmonic functions. When $Z_z = f_1$,

$$\nabla_1^2 Z_z + \frac{\partial^2 Z_z}{\partial z^2} = 0,$$

which gives on averaging

$$\nabla_1^2 \overline{Z}_z + \left(\frac{\partial Z_z}{\partial z}\right)_{z=h} - \left(\frac{\partial Z_z}{\partial z}\right)_{z=-h} = 0.$$

The equations (1c) and (2) show that $\partial Z_z / \partial z$ vanishes on $z = \pm h$. Then the above equation shows that, when $Z_z = f_1$, $\overline{Z}_z$ is a plane harmonic function. Hence, in the general case, we can write

$$\overline{Z}_z = \phi(x, y) + \psi(x, y), \tag{37}$$

where $\phi(x, y)$ is a plane harmonic function and $\psi(x, y)$ the average value of the product of a harmonic function by x or y or z or r^2.

We see that in the problem of the determination of average stresses and displacements in a plate stretched by forces in its plane, a difficulty arises, *viz.*, that the equations involve $\overline{Z}_z$ which is unknown. For a solution of the problem, we are forced to make some plausible assumption about $\overline{Z}_z$, and the solution so obtained gives the average stresses and displacements in the plate, provided an exact solution of the fundamental equations exists which satisfies this assumption. In the case of plane stress ($X_z = Y_z = Z_z = 0$), we know that an exact solution exists and we can say, in this case, that the average stresses and displacements found on this assumption, are the average values of actual stresses and displacements in the plate. When a very thin plate, with its plane faces free from tractions, is stretched by forces in its plane, $\overline{X}_z$, $\overline{Y}_z$, $\overline{Z}_z$, are so small that

they can be neglected. In such a case, the average stresses and displacements in the plate are approximately given by the solution obtained.

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SOLUTION OF A DIOPHANTINE SYSTEM PROPOSED BY BHASKARA

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1. In the 12th century Bhaskara[1] proposed the following problem: To find pairs of integers, $(x\ ;\ y)$, such that $x-y$ is a square and x^2+y^2 is a cube. He gave two numerical solutions, namely (100 ; 75) and (17, 661 ; 16, 820). In this paper other numerical solutions are obtained and general formulas are given which may be used to obtain an infinity of solutions.

The essential problem is to solve the diophantine system:

$$x^2+y^2 = z^3, \tag{1}$$

$$x-y = t^2. \tag{2}$$

The complete solution[2] of (1) can be obtained, but it involves too many parameters to be handled with facility in imposing condition (2). Two sets of solutions of (1) will be considered here.

2. The first solution of (1) which we shall consider is

$$x = r^3-3rs^2, \tag{3}$$

$$y = 3r^2s-s^3. \tag{4}$$

Imposing the condition (2) gives

$$r^3-3rs^2-3r^2s+s^3 = t^2 \tag{5}$$

which has the solution[3]

$$r = u^4+6u^2v^2+8uv^3+21v^4, \tag{6}$$

$$s = -4v(u^3+3u^2v-3uv^2-v^3). \tag{7}$$

3. Equation (5) may be factored,

$$(r+s)(r^2-4rs+s^2) = t^2.$$

[1] *Algebra, with arithmetic and mensuration, from the Sanskrit of Brahmagupta and Bhaskara.* Translation by Henry Thomas Colebrooke, London, 1817.

[2] $x^2+y^2 = z^3$ has the complete solution

$$x = (m^2+n^2)[(mr+ns)(p^2-q^2)+2(ms-nr)pq]$$
$$y = (m^2+n^2)[2(mr+ns)pq-(ms-nr)(p^2-q^2)],$$
$$z = (m^2+n^2)(p^2+q^2)$$

where $p^2+q^2 = r^2+s^2$.

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[3] $r^3+ar^2s+brs^2+cs^3 = z^2$ has the solution

$$r = u^4-2bu^2v^2+8cuv^3+(b^2-4ac)v^4,$$
$$s = -4v(u^3-au^2v+buv^2-cv^3).$$

A. M. Legendre, *Theorie des nombres*, 1808, p. 139. See Dickson, *History of the theory of numbers*, 1920, vol. 2, p. 570.

This last equation has the complete solution[4]

$$r+s = ab^2,$$

$$r^2-4rs+s^2 = ac^2.$$

For simplicity we shall take the special case $a = 1$. Then the second of these becomes

$$r^2-4rs+s^2 = c^2\ ;$$

which has the solution[5]

$$r = q^2-p^2, \tag{8}$$

$$s = 2p(q-2p). \tag{9}$$

Imposing the condition $r+s = b^2$ gives

$$q^2-5p^2+2pq = b^2\ ;$$

which has the solution[5]

$$q = u^2+5v^2,$$

$$p = 2v(u+v).$$

Substitution of these in (8), (9) gives

$$r = u^4+6u^2v^2-8uv^3+21v^4,$$

$$s = 4v(u^3-3u^2v-3uv^2+v^3),$$

which are essentially the same as the values given in equations (6) and (7).

4. Another solution of $x^2+y^2 = z^3$ is

$$x = r(r^2+s^2),$$

$$y = s(r^2+s^2).$$

Imposing the condition $x-y = t^2$ gives

$$(r-s)(r^2+s^2) = t^2,$$

which has the complete solution[4]

$$r-s = kb^2,$$

$$r^2+s^2 = kc^2.$$

The second of these has a solution[6] only when k is of the form $k = m^2+n^2$.

For $k = 1$, this equation becomes

$$r^2+s^2 = c^2,$$

[4] $xy = z^2$ has the complete solution

$$x = af^2, \quad y = ag^2, \quad z = afg,$$

where $(f, g) = 1$.

[5] $x^2+by^2+dxy = z^2$ has the solution

$$x = q^2-bp^2, \quad y = dp^2+2pq.$$

A. Desboves, *Nov. Ann. Math.*, (2) 18 (1879) p. 269 ; (3) 5 (1886) pp. 226-33. See Dikson, *ibid*, vol. 2, p. 405.

[6] $x^2+y^2 = cz^2$ is not solvable unless $c = m^2+n^2$.
See Dickson, *ibid*, vol. 2, p. 405.

which has the complete solution[7]

$$r = 2pqw, \tag{10}$$

$$s = w(p^2-q^2). \tag{11}$$

Imposing the condition $r-s = b^2$ gives

$$2pqw - wp^2 + wq^2 = b^2. \tag{12}$$

Hence $b^2 = wv$, and therefore[4]

$$b = efg, \quad w = ef^2, \quad v = eg^2.$$

For $e = 1$ in the last, (12) becomes

$$q^2+2pq-p^2 = g^2,$$

which has the solution[5]

$$q = u^2+v^2,$$

$$p = 2v(u+v).$$

Substituting these in (10), (11) we get

$$r = 4v(u+v)(u^2+v^2)f^2,$$

$$s = [4v^2(u+v)^2-(u^2+v^2)^2]f^2.$$

Now if $(x; y)$ is any solution of (1), (2), then $(xf^6; yf^6)$ is trivially a solution. Hence the factor f^2 may be discarded :

$$r = 4v(u+v)(u^2+v^2), \tag{13}$$

$$s = 4v^2(u+v)^2-(u^2+v^2)^2. \tag{14}$$

Thus (1), (2) have the solution

$$x = r(r^2+s^2), \tag{15}$$

$$y = s(r^2+s^2), \tag{16}$$

where r, s are given by (13), (14).

5. From equations (3), (4), (6) and (7) we find the following numerical solutions.

(a) $u = 0, \quad v = 0$

$x = 8253 \qquad x^2+y^2 = (457)^3$

$y = 5228 \qquad x-y = (55)^2$

(b) $u = -1, \quad v = 0$

$x = -7360 \qquad x^2+y^2 = (656)^3$

$y = -15, 104 \qquad x-y = (88)^2$

(c) $u = 2, \quad v = 1$

$x = -168, 091 \qquad x^2+y^2 = (8633)^3$

$y = -784, 316 \qquad x-y = (785)^2$

[7] $x^2+y^2 = z^2$ has the complete solution

$$x = 2pqt, \quad y = t(p^2-q^2), \quad z = t(p^2+q^2).$$

See Dickson, *ibid*, vol. 2, p. 169.

$(d)\quad u = 1, \quad v = 2$

$x = 62,975,225 \qquad x^2+y^2 = (191,441)^3$

$y = 55,230,186 \qquad x-y = (2783)^2$

6. Using equations (13), (14), (15) and (16) we find the following numerica solutions.

$(a)\quad u = 0, \quad v = 0$

$x = 100 \qquad x^2+y^2 = (25)^3$

$y = 75 \qquad x-y = (5)^2.$

$(b)\quad u = -2, \quad v = 1$

$x = -16,820 \qquad x^2+y^2 = (841)^3$

$y = -17,661 \qquad x-y = (29)^2$

$(c)\quad u = 1, \quad v = 2$

$x = 3,427,320 \qquad x^2+y^2 = (28,561)^3$

$y = 3,398,759 \qquad x-y = (169)^2$

$(d)\quad u = -1, v = 2$

$x = 67,240 \qquad x^2+y^2 = (1681)^3$

$y = -15,129 \qquad x-y = (287)^2$

$(e)\quad u = 2, \quad v = 1$

$x = 228,260 \qquad x^2+y^2 = (3721)^3$

$y = 40,931 \qquad x-y = (427)^2$

The first two are Bhaskara's solutions.

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ON THE DIFFERENTIABILITY OF AN INDEFINITE INTEGRAL

By

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1. Let $f(x)$ be any bounded and continuous function of x with a discontinuity of the second kind at $x=0$. Let us consider the indefinite integral

$$F(x) = \int_0^x f(x)\,dx.$$

It is a well known problem to investigate the existence of the differential coefficient of $F(x)$ at $x=0$.

Thomae (1893), after some wrong guesses, proved that $F'(0)$ exists provided

$$\lim_{\epsilon \to 0} \int_\epsilon^h \frac{f(x)}{x}\,dx \quad \text{exists.}$$

But it may be pointed out that Thomae's condition* is only sufficient and not necessary.**

Ganesh Prasad (1924) considered a particular case of $f(x)$, namely

$$f(x) = \cos\psi(x),$$

where $\psi(x)$ tends to infinity as x tends to zero. He established that $F'(0)$ exists or not according as

$$\psi(x) \succ \log(1/x^2) \quad \text{or} \quad \psi(x) \preceq \log(1/x^2)$$

respectively.

* It is interesting to note that Thomae's result can be looked upon to be a particular case (for $\delta=1$) of a more general result proved in another connection by B. N. Prasad (1932, p. 188), namely, *if $f(x)$ is integrable in $(0, 1)$*, one has, when $x \to 0$,

$$F(x) = \int_0^x f(t)dt = O(x^\delta),$$

where $\delta > 0$, such that $\int_0^x \frac{f(t)}{t^\delta} dt$ *exists.*

** For consider the function

$$f(x) = \cos[\log(1/x^2)]^{\frac{1}{2}}/[\log(1/x^2)]^{\frac{1}{2}},$$

$$f(x) = 0.$$

This function $f(x)$ is continuous at $x=0$; hence $F'(0)=0$. But Thomae's condition is not satisfied, because

$$\int_\epsilon^h \frac{f(x)}{x}\,dx = [-\sin\{\log(1/x^2)\}^{\frac{1}{2}}]_\epsilon^h$$

which does not exist for $\epsilon \to 0$.

In this paper we shall investigate the same problem for any function $f(x)$ as defined above with the only limitation that $f(x) \geq 0$ for $x \geq 0$. The necessary and sufficient condition, and three other conditions which are sufficient but not necessary, shall be established. Suitable examples shall be constructed to illustrate the various cases.

2. In a previous paper (Shukla, 1945) we have established conditions for the differentiability of monotone functions at a point. If $P(x)$ is any monotone non-decreasing and continuous function of x with $P(0) = 0$, then we have proved in particular the following theorems :—

Theorem 1. *The necessary and sufficient condition for the existence of $P'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{P(x_r)}{x_r} \quad \textit{exists},$$

where $\{x_r\}$ is a sequence of positive values of x converging to zero, such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

COROLLARY. If the above condition is satisfied then

$$P'_+(0) = \lim_{r\to\infty} \frac{P(x_r)}{x_r}.$$

Theorem 2. *The necessary and sufficient condition for the existence of $P'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{P(x_r)}{Q(x_r)} \quad \textit{exists},$$

where $\{x_r\}$ is a sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1,$$

and where $Q(x)$ is any differentiable function of x with $Q(0) = 0$ and $Q'_+(0) \neq 0$.

COROLLARY. If the above condition is satisfied, then

$$P'_+(0) = Q'_+(0).\lim_{r\to\infty} \frac{P(x_r)}{Q(x_r)}.$$

Theorem 3. *A sufficient condition for the existence of $P'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r - x_{r+1}} \quad \textit{exists},$$

where $\{x_r\}$ is any sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

COROLLARY. If the above theorem is satisfied, then

$$P'_+(0) = \lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r - x_{r+1}}.$$

Theorem 4. *$P'_+(0)$ exists and equals zero provided*

$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r-x_{r+1}} = 0,$$

where $\{x_r\}$ is any sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} \neq 0.$$

Theorem 5. *$P'_+(0)$ exists and equals zero provided*

$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_{r+1}} = 0, \tag{1}$$

where $\{x_r\}$ is any sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 0; \tag{2}$$

and provided

$$\frac{P(x_r)-P(x_{r+1})}{x_r-x_{r+1}},$$

which due to (1) *and* (2) *has necessarily to tend to zero for $r\to\infty$, does so monotonically.*

These results have been stated here only for the sake of brevity and convenience. For, as will be seen below, the theorems established in this paper are easily deducible from these results (although they can be proved independently as well).

If $f(x) \geq 0$, then it is observed that the function

$$F(x) = \int_0^x f(x)\,dx$$

is a monotone non-decreasing and continuous function of x with $F(0) = 0$. Hence we can replace the function $P(x)$ there by the function $F(x)$ here in order to obtain the theorems established below.

NECESSARY AND SUFFICIENT CONDITION

3. **Theorem.** *The necessary and sufficient condition for the existence of $F'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{1}{x_r}\int_0^{x_r} f(x)\,dx.$$

exists, where $\{x_r\}$ is a sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

This follows from Theorem 1.

3.1. COROLLARY. If the condition of the above theorem is satisfied, then from the corollary to Theorem 1 it follows that

$$F'_+(0) = \lim_{r\to\infty} \frac{1}{x_r}\int_0^{x_r} f(x)\,dx.$$

4. The following is another form of the necessary and sufficient condition:

Theorem. *The necessary and sufficient condition for the existence of $F'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{1}{\psi(x_r)} \int_0^{x_r} f(x)dx$$

exists, where $\{x_r\}$ is a sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1,$$

and where $\psi(x)$ is any differentiable function of x with $\psi(0) = 0$ and $\psi'_+(0) \neq 0$.

This follows from Theorem 2.

4.1. Corollary. If the condition of the above theorem is satisfied then from the corollary to Theorem 2 it follows that

$$F'_+(0) = \psi'_+(0). \lim_{r\to\infty} \frac{1}{\psi(x_r)} \int_0^{x_r} f(x)dx.$$

Sufficient conditions*

5. In a problem like the present one it is the sufficient conditions which are more useful. Three such conditions shall, therefore, be established below; and it shall be shown that each of them is sufficient and not necessary.

Theorem. *A sufficient condition for the existence of $F'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{\int_{x_{r+1}}^{x_r} f(x)dx}{x_r - x_{r+1}}$$

exists, where $\{x_r\}$ is any sequence of positive values of x converging to zero, such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

This follows from Theorem 3.

Corollary. If the condition of the above theorem is satisfied, then from the corollary to Theorem 3 it follows that

$$F'_+(0) = \lim_{r\to\infty} \frac{\int_{x_{r+1}}^{x_r} f(x)dx}{x_r - x_{r+1}}.$$

5.1. As an application of the above theorem consider the function

$$f(x) = |\cos(1/x)|.$$

If we choose the sequence $\{x_r\}$ to be the sequence of zeros of $\cos(1/x)$ we have $x_r = 2/(2r+1)\pi$, $r = 0, 1, 2, 3, \ldots$, so that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

* For the necessary conditions for the same problem see Shukla, P. D. (1945).

Also it can be shown that

$$\lim_{r\to\infty} \frac{\int_{x_{r+1}}^{x_r} f(x)dx}{x_r - x_{r+1}} = \lim_{r\to\infty} \frac{\int_{2/(2r+3)\pi}^{2/(2r+1)\pi} |\cos(1/x)|\, dx}{[2/(2r+1)\pi - 2/(2r+3)\pi]} = \frac{2}{\pi}.$$

For by putting $x = 1/t$ and applying the Mean Value Theorem of the Integral Calculus (Pierpont, 1905), we have

$$\int_{2/(2r+3)\pi}^{2/(2r+1)\pi} \cos(1/x)dx = \frac{1}{(r\pi + l\pi)^2} \int_{r\pi+\pi/2}^{r\pi+3\pi/2} \cos t\, dt, \quad \tfrac{1}{2} < l < \tfrac{3}{2}$$

$$= \frac{(-)^{r+1}2}{(r\pi + l\pi)^2}$$

so that

$$\lim_{r\to\infty} \frac{\int_{2/(2r+3)\pi}^{2/(2r+1)\pi} |\cos(1/x)|\, dx}{[2/(2r+1)\pi] - [2/(2r+3)\pi]} = \frac{2}{\pi}.$$

Thus the condition of § 5 is satisfied. Consequently $F'_+(0)$ must exist and be equal to $2/\pi$. Also $|\cos(1/x)|$ being an even function of x, its indefinite integral $F(x)$ is an odd function of x, so that $F'_-(0)$ also must exist and be equal to $2/\pi$. That is, $F'(0)$ must exist and be equal to $2/\pi$.

For further confirmation of this result it may be noted with interest that the differential coefficient $F'(0)$ of the function

$$F(x) = \int_0^x \left|\cos\left(\frac{1}{x}\right)\right| dx$$

has already been considered by several writers before. Denjoy (1915) showed it to be $2/\pi$; and Ganesh Prasad (1928 a, b) proved independently that $F'(0)$ does exist and is equal to $2/\pi$.*

5.2. That the condition in § 5 is only sufficient and not necessary can be illustrated by the following example.

In the interval (0, 1) describe a set of points given by $x = 1/r$, $r = 1, 2, 3, \ldots$. Divide each interval $[1/(r+1), 1/r]$ into two subintervals in the ratio $g : g'$, so that we get the subintervals $[1/(r+1),\ 1/(r+1) + g/(g+g')r(r+1)]$ and $[1/(r+1) + g/(g+g')r(r+1),\ (1/r)]$. On these subintervals describe the curve $y = \{\psi(x)\}^2$ where $\psi(x)$ defines the contour of the ellipses described on these subintervals as minor axes with $(k+p)$ and $(k-p)$ to be their semi-major axes $(k > p)$—the semi-major axes are $(k+p)$ on those subintervals which are proportional to g, and $(k-p)$ on those which are proportional to g'.

Obviously this function $f(x)$ is bounded. It is also continuous for all values of x, except for $x = 0$ where it has a discontinuity of the second kind.

* It may be added that, Fejér (1925), obviously being ignorant of Denjoy's result of 1915, asserted in another connection that $F'(0)$ can not exist.

It can be shown† that the integral of $f(x)$ is differentiable at $x = 0$ with

$$F'_{+}(0) = \frac{2[g'(k-p)^2 + g(k+p)^2]}{3(g+g')}.$$

But the condition of § 5 is not satisfied. For if we choose the sequence $\{x_r\}$ to be the sequence of zeros of $f(x)$ we have

$$\lim_{r\to\infty} \frac{1/(r+1)}{1/(r+1)+g/(g+g')r(r+1)} = 1,$$

and

$$\lim_{r\to\infty} \frac{1/(r+1)+g/(g+g')r(r+1)}{1/r} = 1.$$

But

$$\lim_{r\to\infty} \frac{\int_{x_{r+1}}^{x_r} f(x)dx}{x_r - x_{r+1}}$$

does not exist ; for from the definition of $f(x)$ it is seen that this limit is $(k+p)^2$ or $(k-p)^2$ depending on the sequence $\{x_r\}$ approaching the limit zero for $r \to \infty$.

6. The following is another sufficient condition.

Theorem. *$F'_{+}(0)$ exists and equals zero provided*

$$\lim_{r\to\infty} \frac{\int_{x_{r+1}}^{x_r} f(x)dx}{x_r - x_{r+1}} = 0,$$

where $\{x_r\}$ is any sequence of positive values of x converging to zero such that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} \neq 0.$$

This follows from Theorem 4.

6.1. As an application of this result consider the following example.

In the interval (0, 1) describe a set of points given by $x = 1/3^r$, $r = 0, 1, 2, 3, \ldots$. Let any one of these intervals, say $(1/3^{r+1}, 1/3^r)$, be called AB. In the middle of AB place a subinterval CD, such that $CD = (1/r)(1/3^r - 1/3^{r+1})$. On CD as base describe an acute angled triangle CED whose height is 1. Then in the interval $(1/3^{r+1}, 1/3^r)$ let a function $f(x)$ be defined by the broken line $ACEDB$. Let the same be done for all values of r, so that $f(x)$ is completely defined in (0, 1).

Obviously then $f(x)$ is a bounded function of x. It is also continuous in (0, 1) except at $x = 0$ where it has a discontinuity of the second kind.

Let the sequence $\{x_r\}$ be $\{1/3^r\}$ so that

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = \frac{1}{3} \neq 0.$$

† This example has been taken from a paper of Lakshmi Narain (1922 23). The integration image found by him there is our $F'_{+}(0)$ here, with the only difference that his point of discontinuity $x = w$ has been replaced by $x = 0$ here.

And from the definition of $f(x)$ we have

$$\operatorname*{Lim}_{r\to\infty}\frac{\displaystyle\int_{x_{r+1}}^{x_r} f(x)dx}{x_r-x_{r+1}}=\operatorname*{Lim}_{r\to\infty}\frac{(1/2)(1/r)(1/3^r-1/3^{r+1})}{(1/3^r-1/3^{r+1})}=0.$$

Thus the condition of § 6 is fulfilled. Hence $F'_+(0)$ must exist and be equal to zero.

That $F'_+(0)$ exists and equals zero can be proved directly as follows:

Choose any x in (0, 1). It must lie in the interval $(1/3^{r+1}, 1/3^r)$ for some value of r. Suppose it is so for $r=n$, *i.e.*,

$$1/3^{n+1}\leq x\leq 1/3^n.$$

Then for such an x, because of the monotone character of $F(x)$, we have

$$0<\frac{F(x)}{x}<\frac{F(1/3^n)}{1/3^{n+1}}. \tag{3}$$

Now

$$F\left(\frac{1}{3^n}\right)=\int_0^{1/3^n} f(x)dx=\sum_{m=0}^{\infty}\int_{1/3^{n+m+1}}^{1/3^{n+m}} f(x)dx=\sum_{m=0}^{\infty}\frac{1}{(n+m)3^{n+m+1}}<\frac{1}{n.3^{n+1}}\sum_{m=0}^{\infty}\frac{1}{3^m},$$

which is equal to $1/2n3^n$. Thus

$$\operatorname*{Lim}_{n\to\infty}\frac{F(1/3^n)}{1/3^{n+1}}\leq\operatorname*{Lim}_{n\to\infty}\frac{3}{2n},$$

which is equal to 0. Hence from (3) it follows that

$$\operatorname*{Lim}_{x\to 0^+}\frac{F(x)}{x}=0,$$

i.e., $F'_+(0)$ exists and equals zero.

6.2. That the condition given in § 6 is sufficient and not necessary is again proved by the same function $f(x)$ defined in § 5.2. For, although $F'_+(0)$ exists there, yet the condition of § 6 is not satisfied.

7. The following is still another sufficient condition.

Theorem. $F'_+(0)$ *exists and equals zero provided*

$$\operatorname*{Lim}_{r\to\infty}\frac{1}{x_{r+1}}\int_{x_{r+1}}^{x_r} f(x)dx=0, \tag{4}$$

where $\{x_r\}$ *is any sequence of positive values of* x *converging to zero such that*

$$\operatorname*{Lim}_{r\to\infty}\frac{x_{r+1}}{x_r}=0\,; \tag{5}$$

and provided

$$\frac{\displaystyle\int_{x_{r+1}}^{x_r} f(x)dx}{x_r-x_{r+1}},$$

which due to (4) *and* (5) *has necessarily to tend to zero for* $r\to\infty$, *does so monotonically.*

This follows from Theorem 5.

7.1. The function $f(x)$ defined below illustrates an application of the above theorem.

In the interval $(0, 1)$ describe a set of points given by $x = e^{-r(r+1)}$, $r = 0, 1, 2, 3, \ldots$. Let any one of these intervals, say $(e^{-(r+1)(r+2)}, e^{-r(r+1)})$, be called AB. In the middle of AB place a subinterval CD, such that $CD = e^{-(r+1)^2}(e^{-r(r+1)} - e^{-(r+1)(r+2)})$, *i.e.*, $e^{-(r+1)^2}$ times AB. On CD as minor axis describe a semi-ellipse CED whose major semi-axis is $+1$. Let a function $f(x)$ be defined by the contour $ACEDB$. Let the same be done for all values of r, so that $f(x)$ is defined for all values of x in $(0, 1)$.

This function $f(x)$ is a bounded function of x. It is also continuous in $(0, 1)$ except at $x = 0$ where it has a discontinuity of the second kind.

If we choose the sequence $\{x_r\}$ to be such that $x_r = e^{-r(r+1)}$, we have

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 0$$

Also the area of the semi-ellipse with $e^{-(r+1)^2}(e^{-r(r+1)} - e^{-(r+1)(r+2)})$ as minor axis and with 1 as semi-major axis is $\frac{1}{2}\pi e^{-(r+1)^2}(e^{-r(r+1)} - e^{-(r+1)(r+2)})$. Hence

$$\lim_{r\to\infty} \frac{1}{x_{r+1}} \int_{x_{r+1}}^{x_r} f(x)dx = \lim_{r\to\infty} \frac{\pi e^{-(r+1)^2}(e^{-r(r+1)} - e^{-(r+1)(r+2)})}{2e^{-(r+1)(r+2)}} = 0.$$

Thus the condition of § 7 is satisfied, so that $F'_+(0)$ must exist and be equal to zero.

That $F'_+(0)$ exists and equals zero can be seen directly also. For whatever be x in $(0, 1)$, it must be in one of the intervals $(e^{-(r+1)(r+2)}, e^{-r(r+1)})$. Suppose it lies in $(e^{-(n+1)(n+2)}, e^{-n(n+1)})$. Then because of the monotone character of $F(x)$, so long as x lies in $(e^{-(n+1)(n+2)}, e^{-n(n+1)})$, we have

$$0 < \frac{F(x)}{x} < \frac{F(e^{-n(n+1)})}{e^{-(n+1)(n+2)}}. \tag{6}$$

Now,

$$F(e^{-n(n+1)}) = \int_0^{e^{-n(n+1)}} f(x)dx = \sum_{m=0}^{\infty} \int_{e^{-(n+m+1)(n+m+2)}}^{e^{-(n+m)(n+m+1)}} f(x)dx$$

$$= \sum_{m=0}^{\infty} \frac{\pi}{2} e^{-(n+m+1)^2}\left[e^{-(n+m)(n+m+1)} - e^{-(n+m+1)(n+m+2)}\right],$$

$$< \frac{\pi}{2} e^{-(n+m)^2} \sum_{m=0}^{\infty}\left[e^{-(n+m)(n+m+1)} - e^{-(n+m+1)(n+m+2)}\right]$$

which is equal to

$$\frac{\pi}{2} e^{-(n+1)^2 - n(n+1)}.$$

Hence

$$\lim_{n\to\infty} \frac{F(e^{-n(n+1)})}{e^{-(n+1)(n+2)}} \leq \lim_{n\to\infty} \frac{\pi}{2} e^{-(n+1)^2 - n(n+1) + (n+1)(n+2)}$$

which is $= 0$. Thus from (6) it follows that

$$\lim_{x\to 0^+} \frac{F(x)}{x}$$

exists and equals zero. That is, $F'_+(0)$ exists and equals zero.

7.2. To prove that the condition given in § 7 also is sufficient and not necessary, consider the function $f(x)$ defined in § 7.1 with a change in the definition of CD. Let now CD be equal to e^{-2r} times AB, *i.e.*, $CD = e^{-2r}(e^{-r(r+1)} - e^{-(r+1)(r+2)})$. Now,

$$\lim_{r\to\infty} \frac{1}{x_{r+1}} \int_{x_{r+1}}^{x_r} f(x)dx = \lim_{r\to\infty} \frac{\frac{1}{2}\pi e^{-2r}(e^{-r(r+1)} - e^{-(r+1)(r+2)})}{e^{-(r+1)(r+2)}} = \tfrac{1}{2}\pi e^2 \neq 0.$$

Hence the condition of § 7 is not satisfied. But it shall now be shown that $F'_+(0)$ exists and equals zero.

If x is in AB, it must be either in AC, or CD, or DB. Let us consider all the three possibilities.

If x is in AC, then

$$F(x) = \frac{\pi}{2} \sum_{p=1}^{\infty} e^{-2(r+p)}(e^{-(r+p)(r+p+1)} - e^{-(r+p+1)(r+p+2)}),$$

which is

$$< \tfrac{1}{2}\pi\, e^{-2(r+1)}(e^{-(r+1)(r+2)}).$$

Hence

$$\frac{F(x)}{x} < \frac{\frac{1}{2}\pi e^{-2(r+1)-(r+1)(r+2)}}{e^{-(r+1)(r+2)}},$$

which tends to zero for $r\to\infty$. Hence $\lim_{x\to 0^+} [F(x)/x]$ is zero; for $f(x)$ being always ≥ 0, this limit can not be negative.

Similarly if x is in CD or DB it can be proved that $\lim_{x\to 0^+} [F(x)/x]$ again equals zero.

It follows, therefore, that

$$\lim_{x\to 0^+} \frac{F(x)}{x}$$

uniquely exists and equals zero. That is, $F'_+(0)$ exists and equals zero.

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ANHARMONIC PULSATIONS OF A HOMOGENEOUS STAR: EFFECT OF THE RATIO OF SPECIFIC HEATS

BY

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It is well-known that the velocity-time curve for the Cepheid Variables is not sinusoidal: in fact the time interval between the maximum velocity of approach and the maximum velocity of recession is nearly five times larger than the interval between the maximum velocity of recession and the maximum velocity of approach. Again the luminosity of the Cepheids is maximum when the velocity of approach is maximum. In his George Darwin lecture on the pulsation theory of Cepheid Variables Prof. Rosseland (1943) has discarded the assumption of sinusoidal oscillations usually made in theoretical investigations and has developed the general theory of anharmonic oscillations. The actual application of the Rosseland theory to Cepheids is extremely involved mathematically and so far only homogeneous model has been investigated. Rosseland's treatment of the homogeneous model was based on an approximation which significantly affects the physical interpretation of the results. As shown in a recent paper (1944) the anharmonic pulsations of the homogeneous model can be treated exactly for $\gamma = \frac{5}{3}$, γ being the ratio of the two specific heats. The present paper is a continuation of the one mentioned above and here the anharmonic pulsations of a homogeneous model for $\gamma = \frac{3}{2}$ and $\frac{13}{9}$ are considered. These values simplify the integration of the equation of motion.

1. Let r, ρ, p, T and g denote the radius-vector, density, pressure, temperature and gravity of an element in the Lagrangian sense and a, ρ_0, p_0, T_0 and g_0 denote these quantities at a given initial time t_0, which is in the sequel identified with the instant when the pulsation velocity is maximum, it being directed outwards. For a homogeneous star pulsating in the fundamental mode the displacement at any point a is given by

$$r - a = a\eta q, \tag{1}$$

where η is a constant and q a function of time only. For homogeneous model

$$p_0 = \frac{1}{2}\rho_0 \frac{GM}{R}\left(1 - \frac{a^2}{R^2}\right) \tag{2}$$

and

$$g_0 = \frac{GMa}{R^3}, \tag{3}$$

where M is the mass of the star and R its radius at time t_0.

The kinetic energy $W_1(t)$ of radial oscillation is

$$W_1(t) = \frac{1}{2}\int_0^R 4\pi a^2\rho_0 da.a^2\eta^2\dot{q}^2 = \tfrac{3}{10}MR^2\eta^2\dot{q}^2 \tag{4}$$

and the work $W_2(t)$ done against gravitation, measured from the state at time t_0, is

$$W_2(t) = -\frac{3}{5}\frac{GM^2}{R}\left[1-\frac{1}{1+\eta q}\right]. \tag{5}$$

Further the thermal energy $W_3(t)$ measured again from the state at time t_0 is

$$W_3(t) = \int_0^R c_v(T-T_0)4\pi a^2\rho_0 da,$$

where c_v is the constant-volume specific heat per unit mass. Substituting the perfect gas equation

$$p = c_v(\gamma-1)\rho T$$

and the adiabatic relation
$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma = \frac{1}{(1+\eta q)^{3\gamma}},$$

where γ is the ratio of specific heats, we obtain using (2)

$$W_3 = \frac{1}{5(\gamma-1)}\frac{GM^2}{R}\left[\frac{1}{(1+\eta q)^{3\gamma-3}}-1\right]. \tag{6}$$

Adding (4), (5) and (6), we have for the total energy W, which does not vary with time

$$W = \tfrac{3}{5}MR^2\left[\tfrac{1}{2}\eta^2\dot{q}^2 + \frac{GM}{R^3}\left\{1-\frac{1}{1+\eta q}\right\}+\frac{GM}{R^3}\frac{1}{3(\gamma-1)}\left\{\frac{1}{(1+\eta q)^{3\gamma-3}}-1\right\}\right]. \tag{7}$$

It is convenient to introduce a new variable $x(t)$, where Rx represents the displacement at the surface at time t, so that from (1)

$$x = \eta q, \tag{8}$$

Introducing this variable in (7), we get

$$W = \tfrac{3}{5}MR^2\left[\tfrac{1}{2}\dot{x}^2 + \frac{GM}{R^3}\left\{1-\frac{1}{1+x}\right\}+\frac{GM}{R^3}\frac{1}{3(\gamma-1)}\left\{\frac{1}{(1+x)^{3\gamma-3}}-1\right\}\right]. \tag{9}$$

Let x_1 and x_2 represent respectively the 'out-side' and the 'inside' amplitudes of oscillation, *i.e.*, the stellar radius oscillates between $R(1+x_1)$ and $R(1-x_2)$. Let P denote the period, t_1, the part of P during which the radius of the star is greater than R and t_2, the part of P during which the radius is less than R. The 'skewness' δ of oscillation is defined by

$$\delta = t_1/t_2.$$

The equation (9) can be written as

$$\dot{\xi}^2 = \sigma_1^2\left[-A+\frac{2}{\xi}-\frac{2}{3(\gamma-1)}\frac{1}{\xi^{3\gamma-3}}\right], \tag{10}$$

where

$$\xi = 1+x, \quad \sigma_1^2 = \frac{GM}{R^3}$$

and

$$A = \frac{2(3\gamma-4)}{3(\gamma-1)}\left[1-\frac{W}{W_0}\right],$$

$(-W_0)$ being the total energy of the static configuration of radius R. Differentiation of (10) with respect to time gives the equation of motion

$$\ddot{\xi} = \sigma_1^2\left[\frac{1}{\xi^{3\gamma-2}}-\frac{1}{\xi^2}\right]. \tag{11}$$

2. We shall now consider the equation (10) for some values of γ.

CASE I. $\gamma = \frac{5}{3}$:

$$P = \frac{2\pi}{\sigma_1}\frac{(1+x_1)^3}{(1+2x_1)^{3/2}}, \qquad x_2 = \frac{x_1}{1+2x_1},$$

$$\delta = \frac{\dfrac{x_1(1+x_1)}{1+2x_1}+\dfrac{(1+x_1)^3}{(1+2x_1)^{3/2}}\left[\dfrac{\pi}{2}+\sin^{-1}\dfrac{x_1}{1+x_1}\right]}{\dfrac{-x_1(1+x_1)}{1+2x_1}+\dfrac{(1+x_1)^3}{(1+2x_1)^{3/2}}\left[\dfrac{\pi}{2}-\sin^{-1}\dfrac{x_1}{1+x_1}\right]}.$$

When x_1 is small

$$P \sim \frac{2\pi}{\sigma_1}(1+\tfrac{3}{2}x_1^2), \quad x_2 \sim x_1(1-2x_1).$$

CASE II. $\gamma = \frac{4}{3}$: For $\gamma = \frac{4}{3}$, the equation (10) becomes

$$\dot{\xi}^2 = \sigma_1^2\left[-A+\frac{2}{\xi}-\frac{4}{3}\frac{1}{\xi^{3/2}}\right]$$

or

$$dt = \frac{1}{\sigma_1}\frac{\xi^{3/4}d\xi}{[-A\xi^{3/2}+2\xi^{1/2}-\frac{4}{3}]^{1/2}} = \frac{2}{\sigma_1}\frac{y^{5/2}dy}{\sqrt{\{Q(y)\}}},$$

where

$$Q(y) = -Ay^3+2y-\tfrac{4}{3}$$

and $y = \sqrt{\xi}$.

Let y_1 and y_2 $(y_1 > y_2)$ be the positive roots of $Q(y) = 0$, then

$$P = \frac{4}{\sigma_1}\int_{y_2}^{y_1}\frac{y^{5/2}dy}{\sqrt{Q(y)}} = \frac{2}{\sigma_1 A}\left[\left\{-y^{1/2}\sqrt{Q(y)}\right\}_{y_2}^{y_1}+2\int_{y_2}^{y_1}\frac{y^{1/2}dy}{\sqrt{Q(y)}}-\frac{2}{3}\int_{y_2}^{y_1}\frac{dy}{\sqrt{\{yQ(y)\}}}\right.$$

$$= \frac{8}{\sigma_1 A^{3/2}}\frac{1}{\{y_1(y_1+2y_2)\}^{1/2}}\left[y_2\Pi\left(\frac{\pi}{2}, \left\{\frac{y_1^2-y_2^2}{y_1(y_1+2y_2)}\right\}^{\frac{1}{2}}, -\frac{y_1-y_2}{y_1}\right)\right.$$

$$\left.-\frac{1}{3}F\left(\frac{\pi}{2}, \left\{\frac{y_1^2-y_2^2}{y_1(y_1+2y_2)}\right\}^{\frac{1}{2}}\right)\right],$$

$$t_2 = \frac{4}{\sigma_1}\int_{y_2}^{1} \frac{y^{5/2}}{\sqrt{Q(y)}} = \frac{8}{\sigma_1 A^{3/2}}\,\frac{1}{[y_1(y_1+2y_2)]^{1/2}}\Bigg[y_2\Pi\left(\phi,\ \sqrt{\left\{\frac{y_1^2-y_2^2}{y_1(y_1+2y_2)}\right\}},\ -\frac{y_1-y_2}{y_1}\right)$$

$$-\frac{1}{3}F\left(\phi,\ \sqrt{\left\{\frac{y_1^2-y_2^2}{y_1(y_1+2y_2)}\right\}}\right)-\frac{1}{4}\left\{(\tfrac{2}{3}-A)Ay_1(y_1+2y_2)\right\}^{\frac{1}{2}}\Bigg],$$

and

$$t_1 = P-t_2, \quad \delta = \frac{P-t_2}{t_1},$$

where

$$A = \frac{2(3y_1-2)}{3y_1^3}, \quad y_2 = \frac{y_1}{2}\left[\left\{1+\frac{8}{3y_1-2}\right\}^{\frac{1}{2}}-1\right]$$

and

$$\tan\phi = \left[\frac{y_1}{y_2}\,\frac{1-y_2}{y_1-1}\right]^{\frac{1}{2}}.$$

When x_1 is small

$$P \simeq \frac{2\pi}{\sigma_1}\sqrt{2}.(1+\tfrac{59}{48}x_1^2), \quad x_2 \sim x_1(1-\tfrac{11}{6}x_1).$$

CASE III. $\gamma = \frac{13}{9}$: For $\gamma = \frac{13}{9}$, the equation (10) becomes

$$\dot{\xi}^2 = \sigma_1^2\left[-A+\frac{2}{\xi}-\frac{3}{2}\,\frac{1}{\xi^{4/3}}\right]$$

or

$$dt = \frac{1}{\sigma_1}\,\frac{\xi^{2/3}d\xi}{[-A\xi^{4/3}+2\xi^{1/3}-\frac{3}{2}]^{1/2}} = \frac{3}{\sigma_1}\,\frac{y^4dy}{\sqrt{Q(y)}},$$

where

$$Q(y) = -Ay^4+2y-\tfrac{3}{2}$$

and $y = \xi^{1/3}$.

Let y_1 and $y_2\,(y_1 > y_2)$ be the real positive roots of $Q(y) = 0$, then

$$P = \frac{6}{\sigma_1}\int_{y_2}^{y_1}\frac{y^4dy}{\sqrt{Q(y)}} = \left[\left\{-y\sqrt{Q(y)}\right\}_{y_2}^{y_1}+3\int_{y_2}^{y_1}\frac{ydy}{\sqrt{Q(y)}}-\frac{3}{2}\int_{y_2}^{y_1}\frac{dy}{\sqrt{Q(y)}}\right]$$

$$= \frac{12}{\sigma_1}\,\frac{(p-q)\alpha\beta}{A\sqrt{Q(p)}.\ \sqrt{(\alpha^2+\beta^2)}}\left[\frac{p-q}{1-\alpha^2}\,\Pi\left(\frac{\pi}{2},\ \frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}},\ \frac{\alpha^2}{1-\alpha^2}\right)-(\tfrac{1}{2}-q)\,F\left(\frac{\pi}{2},\ \frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}}\right)\right],$$

$$t_2 = \frac{6}{\sigma_1}\int_{y_2}^{1}\frac{y^4dy}{\sqrt{Q(y)}}$$

$$= \frac{6}{\sigma_1}\,\frac{(p-q)\alpha\beta}{A\sqrt{Q(p)}\ \sqrt{(\alpha^2+\beta^2)}}\Bigg[\frac{p-q}{1-\alpha^2}\,\Pi\left(\theta,\ \frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}},\ \frac{\alpha^2}{1-\alpha^2}\right)-(\tfrac{1}{2}-q)\,F\left(\theta,\ \frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}}\right)$$

$$-(p-q)\sqrt{\left(\frac{\alpha^2+\beta^2}{(1-\alpha^2)(1+\beta^2)}\right)}\times\tan^{-1}\left(\frac{1+\beta^2}{1-\alpha^2}\,\frac{1-t^2}{\beta^2/\alpha^2+t^2}\right)^{\frac{1}{2}}-\sqrt{(\tfrac{1}{2}-A)}\,\frac{\sqrt{Q(p)}.\ \sqrt{(\alpha^2+\beta^2)}}{3(p-q)\alpha\beta}\Bigg]$$

and

$$t_1 = P-t_2, \quad \delta = \frac{P-t_2}{t_1},$$

where

$$A = \frac{4y_1-8}{2y_1^4}, \qquad y_2^3+y_1y_2^2+y_1^2y_2 = \frac{8y_1^3}{4y_1-8}, \qquad p+q = -\frac{y_1^2+y_2^2}{y_1+y_2},$$

$$p-q = \left[3(y_1+y_2)^2-4y_1y_2+\frac{4y_1^2y_2^2}{(y_1+y_2)^2}\right]^{\frac{1}{2}}, \qquad \alpha^2 = \frac{(y_1-p)(p-y_2)}{(y_1-q)(y_2-q)},$$

$$\beta^2 = \frac{\{p+\frac{1}{2}(y_1+y_2)\}^2+\frac{3}{4}(y_1+y_2)^2-y_1y_2}{\{q+\frac{1}{2}(y_1+y_2)\}^2+\frac{3}{4}(y_1+y_2)^2-y_1y_2}, \qquad t = \frac{1}{\alpha}\,\frac{p-1}{1-q}, \qquad \theta = \cos^{-1} t.$$

When x_1 is small

$$P \sim \frac{2\pi}{\sigma_1}\sqrt{3}.(1+\tfrac{85}{72}x_1^2), \qquad x_2 \sim x_1(1-\tfrac{16}{9}x_1).$$

CASE IV. $\gamma = \frac{4}{3}$: For $\gamma = \frac{4}{3}$, equations (10) and (11) give

$$\dot{\xi}^2 = -A\sigma_1^2 = -\text{ve}$$

and

$$\ddot{\xi} = 0.$$

In this case no oscillatory motion is possible and the star is in neutral equilibrium.
In the following numerical calculations x_1 has been taken equal to $\frac{1}{8}$.

γ	x_2	$\frac{P}{(2\pi/\sigma_1)}$	δ
$\frac{5}{3} = 1{\cdot}67$	0·100	1·019	1·33
$\frac{3}{2} = 1{\cdot}50$	0·102	1·436	1·29
$\frac{13}{9} = 1{\cdot}44$	0·103	1·757	1·25

3. For a given outside amplitude, as γ decreases, P and x_2 increase but δ decreases. By suitably choosing γ, for a given outside amplitude, the difference between the observed period of a star and theoretically calculated period on the basis of homogeneous model can be made to vanish. But as we decrease γ. δ also decreases and hence the skewness remains unexplained.

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FINITE STRAIN IN AELOTROPIC ELASTIC BODIES—II

By

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In Part I (Seth, 1945) we have developed the theory of finite strain in aelotropic elastic bodies and have applied it to the cases of uniform tension, hydrostatic pressure and a rectangular plate bent by terminal couples. In this paper the theory has been extended to the following cases:

1. Torsion of a circular cylinder.
2. A spherical shell under uniform normal traction.
3. A cylindrical shell under uniform normal traction.

1. Torsion of a circular cylinder

We assume that the aelotropy is of the hexagonal or rhombohedral type in which the elastic constants are connected by the relation $c_{12} = c_{11} - 2c_{66}$.

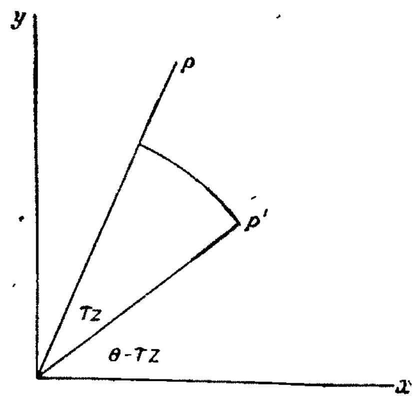

FIG. 1

Let the cylinder be subjected to a finite twist τ. From considerations of symmetry we can take the cross-sections to remain plain and the straight radii to remain straight (Fig. 1). The components of displacement are now seen to be

$$u = x(1-\beta\cos\tau z) - y\beta\sin\tau z, \tag{1.1}$$

$$v = y(1-\beta\cos\tau z) + x\beta\sin\tau z, \tag{1.2}$$

$$w = \alpha z, \tag{1.3}$$

where β is a function of $r = (x^2+y^2)^{\frac{1}{2}}$ only and α is a constant.

The corresponding strain components are

$$s_x = \tfrac{1}{2}(1-\beta^2) - \tfrac{1}{2}x^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \tag{2.1}$$

$$s_y = \tfrac{1}{2}(1-\beta^2) - \tfrac{1}{2}y^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \tag{2.2}$$

$$s_z = \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2, \tag{2.3}$$

$$\sigma_{yz} = \tau\beta^2 x, \quad \sigma_{zx} = -\tau\beta^2 y, \tag{2.4}$$

$$\sigma_{xy} = -xy\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \tag{2.5}$$

where $\beta' = d\beta/dr$.

In cylindrical co-ordinates the stress-components are

$$\widehat{rr} = c_{11}[1-\beta^2 - \tfrac{1}{2}(r^2\beta'^2 + 2r\beta\beta')] + c_{13}(\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2) - c_{66}(1-\beta^2), \tag{3.1}$$

$$\widehat{\theta\theta} = (c_{11} - 2c_{66})[1-\beta^2 - \tfrac{1}{2}(r^2\beta'^2 + 2r\beta\beta')] + c_{13}(\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2) + c_{66}(1-\beta^2), \tag{3.2}$$

$$\widehat{zz} = c_{13}[1-\beta^2 - \tfrac{1}{2}(r^2\beta'^2 + 2r\beta\beta')] + c_{33}(\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2\tau^2 r^2), \tag{3.3}$$

$$\widehat{r\theta} = 0, \quad \widehat{rz} = 0, \quad \widehat{\theta z} = c_{44}\tau r\beta^2. \tag{3.4}$$

All the body-stress equations are identically satisfied excepting

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = 0,$$

which gives

$$(2c_{11} + c_{13}\tau^2 r^2)\beta^2 + c_{11}(r^2\beta'^2 + 2r\beta\beta') + 2c_{66}\int r\beta'^2 dr = k, \tag{4.1}$$

k being a constant.

Putting

$$2c_{66}/c_{11} = c_0, \quad c_{13}/c_{11} = b_0, \quad b_0\tau^2 r^2 = t,$$

we get

$$(1+t)\beta^2 + \left(2t\frac{d\beta}{dt} + \beta\right)^2 + 2c_0\int t\left(\frac{d\beta}{dt}\right)^2 dt = k_1, \tag{4.2}$$

which is of the same form as we get in the isotropic case (Seth, 1935). This differential equation has been already discussed in detail. Its solution, which is absolutely convergent for at least $t \leq 1$, has been found to be

$$\beta_0 = A_0\left[1 - \tfrac{1}{8}t + \frac{6-c_0}{12.8^2}t^2 - \frac{c_0^2}{9.8^4}t^3 + \ldots\right], \tag{5}$$

A_0 being a constant which is to be determined from the boundary conditions.

The boundary $r = a$ is free from traction, and hence $\widehat{rr} = 0$ over $r = a$. This gives

$$(2+b_0) - c_0 - b_0(1-\alpha)^2 = (a\beta_0' + \beta_0)^2 + \beta_0^2(1 - c_0 + b_0\tau^2 a^2), \tag{6}$$

where β_0 is the value of β over $r = a$.

The other boundary condition shows that the tractions on any cross-section are statically equivalent to a single couple whose axis is the z-axis.

Thus

$$\iint_S \widehat{xz}\, dxdy = 0, \quad \iint_S \widehat{yz}\, dxdy = 0, \quad \iint_S \widehat{zz}\, dxdy = 0, \tag{7.1}$$

$$\iint_S y.\widehat{zz}\, dxdy = 0, \quad \iint_S -x.\widehat{zz}\, dxdy = 0. \tag{7.2}$$

$$\int_0^a 2\pi r.\widehat{zz} dr = 0. \tag{7.3}$$

All the conditions in (7.1) and (7.2) are satisfied excepting (7.3), which gives

$$2d_0 + 1 - (1-\alpha)^2 = \frac{2}{a^2}\int_0^a [d_0(r\beta' + \beta)^2 + \beta^2(d_0 + \tau^2 r^2)]r dr, \tag{8}$$

where $d_0 = c_{13}/c_{33}$.

We see that (6) and (8) are both necessary and sufficient to determine the unknown constants A_0 and α.

Proceeding as in the isotropic case (Seth, 1935) we get the equation to determine A_0 as

$$2 - c_0 - 2b_0 d_0 = A_0^2(2 - c_0 - 2b_0 d_0)[1 - \tfrac{1}{4}t_0 + \tfrac{1}{24}(1 - \tfrac{1}{16}c_0)t_0^2 + \ldots) \tag{9.1}$$

or

$$A_0 = 1 + \tfrac{1}{8}t_0 + \tfrac{1}{384}(1 + c_0)t_0^2 + \ldots \tag{9.2}$$

For α we get

$$b_0[1 - (1-\alpha)^2] = \tfrac{1}{4}t_0 + \tfrac{1}{24}t_0^2 + \ldots \tag{9.3}$$

or

$$\alpha = \tfrac{1}{4}\tau^2 a^2 + \tfrac{1}{96}(3 + 2b_0)\tau^4 a^4 + \ldots \tag{9.4}$$

The torsional couple N is given by

$$N = 2\pi c_{44}\tau \int_0^a \beta^2 r^3 dr = \tfrac{1}{2}\pi c_{44}\tau a^4\left[1 + \tfrac{1}{12}b_0\tau^2 a^2 - \frac{b_0^2}{12.16}(1 - \tfrac{1}{4}c_0)\tau^4 a^4 + \ldots\right]. \tag{10}$$

For Beryl, a hexagonal crystal, the constants are

$$c_{11} = 2746, \quad c_{33} = 2409, \quad c_{12} = 980, \quad c_{13} = 674, \quad c_{44} = 666.$$

The rigidity $c_{44} = 666$ is less than that for steel, and $b_0 = c_{13}/c_{11} = 0.245$ is also less than that for steel if we take its Poisson's ratio as 0.31. Thus the couple required to twist a material of the Beryl type through a given angle is less than that required for one of steel of the same dimensions.

For $c_{13} = 0\,(b_0 = 0)$ we get the case discussed by Saint-Venant for large torsional shifts (Saint-Venant, 1855).

2. Spherical shell

We assume there is transverse isotropy about the radius vector. This is a case of curvilinear aelotropy discussed by Saint-Venant when the strain is small (Saint-Venant, 1865).

We take the components of displacement in spherical polar co-ordinates as

$$u_r = r(1 - P), \quad u_\theta = 0, \quad u_\phi = 0, \tag{11}$$

where P is a function of r only.

The strain components are given by

$$s_r = \frac{1}{2}\left[1-\left(1-\frac{du_r}{dr}\right)^2\right], \tag{12.1}$$

$$s_\theta = s_\phi = \frac{1}{2}\left[1-\left(1-\frac{u_r}{r}\right)^2\right], \tag{12.2}$$

$$\sigma_{r\theta} = 0, \quad \sigma_{\theta\phi} = 0, \quad \sigma_{r\phi} = 0. \tag{12.3}$$

The stress-strain relations are of the form

$$\widehat{rr} = c_{33}s_r + c_{13}(s_\theta + s_\phi) = c_{33}s_r + 2c_{13}s_\theta, \tag{13.1}$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = c_{11}s_\theta + (c_{11}-2c_{66})s_\phi + c_{13}s_r = 2(c_{11}-c_{66})s_\theta + c_{13}s_r, \tag{13.2}$$

$$\widehat{\theta\phi} = 0, \quad \widehat{\phi r} = 0, \quad \widehat{r\theta} = 0. \tag{13.3}$$

All the body-stress equations are identically satisfied excepting

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{2}{r}(\widehat{rr}-\widehat{\theta\theta}) = 0,$$

which gives

$$\left(P+r\frac{dP}{dr}\right)\frac{d^2P}{dr^2}+\left(3-\frac{c_{13}}{c_{33}}\right)\left(\frac{dP}{dr}\right)^2+\frac{4P}{r}\frac{dP}{dr}-\frac{1-P^2}{r^2}\left(1+\frac{c_{13}}{c_{33}}-2\frac{c_{11}-c_{66}}{c_{33}}\right) = 0. \tag{14}$$

If we asume the relations

$$2c_{11} = c_{33} + c_{13} + 2_{66},$$

and put

$$2\left(3-\frac{c_{13}}{c_{33}}\right) = F, \qquad \frac{r}{P}\frac{dP}{dr} = V, \tag{15}$$

we get

$$(1+V)P\frac{dV}{dP} + V^2 + 2FV + 3 = 0, \tag{16}$$

which is of the same form as we get in the isotropic case (Sheppherd and Seth, 1936). Various cases of thick and thin shells subjected to uniform, but not necessarily equal, normal tractions on the inner and outer surfaces can now be discussed.

3. Cylindrical shell

Assuming that there is symmetry about the z-axis we get the stress-strain relations to be of the type

$$\widehat{rr} = c_{11}s_r + (c_{11}-2c_{66})s_\theta + c_{13}s_z, \tag{17.1}$$

$$\widehat{\theta\theta} = (c_{11}-2c_{66})s_r + c_{11}s_\theta + c_{13}s_z, \tag{17.2}$$

$$\widehat{zz} = c_{13}(s_r + s_\theta) + c_{33}s_z. \tag{17.3}$$

Taking the displacements in cylindrical co-ordinates as

$$u_r = r(1-Q), \quad u_\theta = 0, \quad u_z = \alpha z, \tag{18}$$

Q being a function r only, we get

$$s_r = \frac{1}{2}\left[1-\left(1-\frac{du_r}{dr}\right)^2\right], \tag{19.1}$$

$$s_\theta = \frac{1}{2}\left[1-\left(1-\frac{u_r}{r}\right)^2\right], \tag{19.2}$$

$$s_z = \alpha - \tfrac{1}{2}\alpha^2. \tag{19.3}$$

The only body-stress equation which is not identically satisfied is

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = 0,$$

which gives

$$\left(Q + r\frac{dQ}{dr}\right)\frac{d^2Q}{dr^2} + \left(2 + \frac{c_{66}}{c_{11}}\right)\left(\frac{dQ}{dr}\right)^2 + \frac{3Q}{r}\frac{dQ}{dr} = 0. \tag{20}$$

Putting

$$\frac{r}{Q}\frac{dQ}{dr} = U, \qquad 2 + c_{66}/c_{11} = 2G,$$

we get

$$(1+U)Q\frac{dU}{dQ} + U^2 + 2GU + 2 = 0, \tag{21}$$

which is of the same form as in the isotropic case. Its solution is

$$\log Q = \frac{G-1}{\sqrt{(2-G^2)}} \tan^{-1}\left(\frac{G+U}{\sqrt{(2-G^2)}}\right) - \tfrac{1}{2}\log\left[(U+G)^2 + 2 - G^2\right] + L, \tag{22}$$

L being a constant.

Various cases of thin and thick cylindrical shells subjected to uniform, but not necessarily equal normal tractions on the inner and outer surfaces can now be discussed.

4. Case of transverse isotropy

When there is transverse isotropy about the radius vector (Saint-Venant, 1865), the stress-strain relations are of the form

$$\widehat{rr} = c_{13}(s_\theta + s_z) + c_{33}s_r, \tag{23.1}$$

$$\widehat{\theta\theta} = c_{11}s_\theta + (c_{11} - 2c_{66})s_z + c_{13}s_r, \tag{23.2}$$

$$\widehat{zz} = (c_{11} - 2c_{66})s_\theta + c_{11}s_z + c_{13}s_r. \tag{23.3}$$

The corresponding equation satisfied by Q is of the form

$$\left(Q+r\frac{dQ}{dr}\right)\frac{d^2Q}{dr^2}+\frac{1}{2}\left(5-\frac{c_{13}}{c_{33}}\right)\left(\frac{dQ}{dr}\right)^2+\frac{3Q}{r}\frac{dQ}{dr}$$

$$-\frac{1}{2r}(1-Q^2)(c_{33}-c_{11})-\frac{1}{r}(\alpha-\tfrac{1}{2}\alpha^2)(c_{13}-c_{11}+2c_{66}) = 0. \qquad (24)$$

This reduces to an equation of the type given in (20) if $\alpha = 0$, and $c_{11} = c_{33}$.

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ON THE CONCEPT OF GENERALIZED PLANE STRESS

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Introduction

In the absence of the knowledge of an exact solution of the equations of equilibrium of a plate whose plane faces are free from tractions, practically useful information about the state of stress in the plate can be obtained if we can determine the average values of the stress components taken over the thickness of the plate. It has been pointed out in a recent note (Ghosh, 1946, p. 10) that it is necessary for the determination of the average stresses in a plate (whose middle plane is taken as the plane $z = 0$) to know beforehand the average value $\bar{Z}_z$ of the stress component Z_z.

Several assumptions can be made about $\bar{Z}_z$, and the solutions so obtained give the average stresses in the plate provided the assumptions are compatible with the fundamental equations of the plate. The simplest assumption that we can make is that of plane stress, in which the stress components X_z, Y_z, Z_z are identically zero throughout the plate. We know that such a state of stress is possible in a plate of an isotropic material, and we get on this assumption the average stresses in the plate when the plate is stretched by arbitrary forces in its plane.

As a previous knowledge of $\bar{Z}_z$ only is required in the average-stress problem, Filon (1903) has suggested some relaxations of the restrictions imposed in the theory of plane stress. He assumes that Z_z is identically zero throughout the plate, while X_z, Y_z are not. Such a state of stress is called by Filon a state of generalized plane stress. Except in a particular case of the bending of a plate given by Love (1927, p. 473), it has not been shown that a state of generalized plane stress is compatible with the fundamental equations of the plate. Southwell (1936, p. 209) has given a solution of the stress equations of equilibrium of a plate which also satisfies the conditions of compatibility of stresses and the condition $Z_z = 0$, but he has not taken account of the conditions that the plane faces of the plate are free from tractions. Assuming such a state of stress to be possible, Southwell discusses its claim to greater generality than that of the state of plane stress. He arrives at the conclusion that from the practical point of view, the state of generalized plane stress is more general than the state of plane stress only in the case of flexural actions, the generality in the case of extensional actions being of a severely restricted type which demand relations between the tractions X_ν, Y_ν, Z_ν, on the rim of the plate, not likely to be satisfied in practice.

In the present paper an exact solution of the stress equations of equilibrium has been obtained which satisfies the stress compatibility relations as well as the boundary

conditions on the plane faces of the plate, thus proving the possibility of a state of generalized plane stress in the plate. The appropriate surface tractions which are to be applied to the cylindrical edge of the plate to maintain this state of generalized plane stress have been calculated. An examination of the expressions for these surface tractions mainly confirms Southwell's conclusions. They show that the state of generalized plane stress is really more general than the state of plane stress when the plate is bent by forces acting on its rim. This conclusion has been arrived at by Love (1927, p. 474) by assuming that on the edge of the plate Z_ν has a parabolic distribution and X_ν, Y_ν linear distributions with respect to z, but his solution is otherwise sufficiently general to admit of the satisfaction of boundary conditions with arbitrarily prescribed values of the stress resultants and the stress couples on the edge line of the plate. These expressions for the surface tractions further show that a slight generality in the stress system can be introduced both in the extension and in the flexure of a plate, in that a restricted variation of edge-tractions with z can be permitted.

Fundamental equations

Let us consider the equilibrium of a plate of an isotropic material, of thickness $2h$, stretched and bent by forces applied to its cylindrical edge. Let us take the middle plane of the plate as the plane $z = 0$.

When there are no body forces, the stress equations of equilibrium are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z} = 0, \tag{1a}$$

$$\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z} = 0, \tag{1b}$$

$$\frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z} = 0. \tag{1c}$$

As the faces $z = \pm h$ of the plate are free from tractions, we have

$$X_z = Y_z = Z_z = 0, \tag{2}$$

when $z = \pm h$. If X_ν, Y_ν, Z_ν are the components of the surface tractions applied to the cylindrical edge of the plate,

$$lX_x+mX_y = X_\nu, \quad lX_y+mY_y = Y_\nu, \quad lX_z+mY_z = Z_\nu, \tag{3}$$

on the edge, where l, m, 0 are the direction cosines of the normal to it. The stress components must also satisfy the compatibility conditions

$$\nabla^2X_x+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = 0, \qquad \nabla^2Y_z+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\partial z} = 0, \tag{4a}$$

$$\nabla^2Y_y+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} = 0, \qquad \nabla^2X_z+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\partial z} = 0, \tag{4b}$$

$$\nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = 0, \qquad \nabla^2 X_y + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\partial y} = 0, \tag{4c}$$

where

$$\Theta = X_x + Y_y + Z_z, \tag{5}$$

and satisfies the equation

$$\nabla^2\Theta = 0. \tag{6}$$

Method of determining the stresses

Let us try to obtain a solution of the fundamental equations with the additional condition,

$$Z_z = 0, \tag{7}$$

throughout the plate. Substituting from (7) in the first equation of (4c), we get

$$\frac{\partial^2\Theta}{\partial z^2} = 0,$$

which gives

$$\Theta = \Theta_0 + z\Theta_1, \tag{8}$$

where Θ_0 and Θ_1 are independent of z. As Θ satisfies the equation (6), Θ_0 and Θ_1 are plane harmonic functions in x, y.

Substitution from (8) in the equations (4) gives

$$\nabla^2 X_x + \frac{1}{1+\sigma}\left\{\frac{\partial^2\Theta_0}{\partial x^2} + z\frac{\partial^2\Theta_1}{\partial x^2}\right\} = 0, \qquad \nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial\Theta_1}{\partial y} = 0, \tag{9a}$$

$$\nabla^2 Y_y + \frac{1}{1+\sigma}\left\{\frac{\partial^2\Theta_0}{\partial y^2} + z\frac{\partial^2\Theta_1}{\partial y^2}\right\} = 0, \qquad \nabla^2 X_z + \frac{1}{1+\sigma}\frac{\partial\Theta_1}{\partial x} = 0, \tag{9b}$$

$$\nabla^2 X_y + \frac{1}{1+\sigma}\left\{\frac{\partial^2\Theta_0}{\partial x\partial y} + z\frac{\partial^2\Theta_1}{\partial x\partial y}\right\} = 0; \tag{9c}$$

the first equation of (4c) having been already disposed of in determining Θ. Since Θ_1 is a plane harmonic function in x, y, we can write the second equation of (9a) and the second equation of (9b) as

$$\nabla^2\left[Y_z - \frac{h^2-z^2}{2(1+\sigma)}\frac{\partial\Theta_1}{\partial y}\right] = 0, \quad \nabla^2\left[X_z - \frac{h^2-z^2}{2(1+\sigma)}\frac{\partial\Theta_1}{\partial x}\right] = 0. \tag{10}$$

Putting $Z_z = 0$ in the equation (1c), and remembering that Θ_1 is a plane harmonic function in x, y, this equation can be written as

$$\frac{\partial}{\partial x}\left[X_z - \frac{h^2-z^2}{2(1+\sigma)}\frac{\partial\Theta_1}{\partial x}\right] + \frac{\partial}{\partial y}\left[Y_z - \frac{h^2-z^2}{2(1+\sigma)}\frac{\partial\Theta_1}{\partial y}\right] = 0. \tag{11}$$

This equation shows that a function ϕ of x, y, z exists such that

$$X_z = \frac{h^2-z^2}{2(1+\sigma)}\cdot\frac{\partial\Theta_1}{\partial x} - \frac{\partial\phi}{\partial y}, \qquad Y_z = \frac{h^2-z^2}{2(1+\sigma)}\cdot\frac{\partial\Theta_1}{\partial y} + \frac{\partial\phi}{\partial x}. \tag{12}$$

If we substitute from (12) in (10), we see that ϕ satisfies the equations

$$\frac{\partial}{\partial x}\nabla^2\phi = 0, \quad \frac{\partial}{\partial y}\nabla^2\phi = 0,$$

which show that $\nabla^2\phi$ is a function of z alone, $F'''(z)$, say. If we put

$$\phi = \frac{\partial^3\psi}{\partial x\partial y\partial z}, \tag{13}$$

we see that ψ satisfies the equation

$$\frac{\partial^3}{\partial x\partial y\partial z}\nabla^2\psi = F'''(z). \tag{14}$$

This equation gives

$$\nabla^2\psi = xyF''(z) + F_1(y, z) + F_2(z, x) + F_3(x, y).$$

where F_1, F_2, F_3 are arbitrary functions of the arguments involved. Let a particular solution of this equation be

$$\psi = xyF(z) + f_1(y, z) + f_2(z, x) + f_3(x, y). \tag{15}$$

It is seen at once that this particular solution does not contribute to the values of X_z, Y_z. Hence we can take

$$\nabla^2\psi = 0. \tag{16}$$

After expressing X_z, Y_z in terms of the functions Θ_1 and ψ by means of the formulae (12) and (13), let us determine suitable expressions for X_x, Y_y, X_y. From the equations (1a), (1b), (12) and (13), we get

$$\frac{\partial}{\partial x}\left[X_x - \frac{z}{1+\sigma}\Theta_1 - \frac{\partial^4\psi}{\partial y^2\partial z^2}\right] + \frac{\partial X_y}{\partial y} = 0,$$

$$\frac{\partial}{\partial y}\left[Y_y - \frac{z}{1+\sigma}\Theta_1 + \frac{\partial^4\psi}{\partial x^2\partial z^2}\right] + \frac{\partial X_y}{\partial x} = 0,$$

As ψ makes its appearance in these equations through X_z, Y_z, and as the particular solution (15) contributes nothing to X_z. Y_z, it also contributes nothing to these equations. Therefore the relevant part of ψ which affects these equations is that given by the equation (16). These equations show that a function χ' of x, y, z exists such that

$$X_x = \frac{\partial^2\chi'}{\partial y^2} + \frac{z}{1+\sigma}\Theta_1 + \frac{\partial^4\psi}{\partial y^2\partial z^2}, \tag{17a}$$

$$Y_y = \frac{\partial^2\chi'}{\partial x^2} + \frac{z}{1+\sigma}\Theta_1 - \frac{\partial^4\psi}{\partial x^2\partial z^2}, \tag{17b}$$

$$X_y = -\frac{\partial^2\chi'}{\partial x\partial y}. \tag{17c}$$

Substituting from the equations (17) in the first equation of (9a), the first equation of (9b) and the equation (9c), and remembering that ψ is a harmonic function in x, y, z, and Θ_0, Θ_1 plane harmonic functions in x, y, we get

$$\frac{\partial^2}{\partial x^2}\left[\nabla^2\chi'-\frac{1}{1+\sigma}\Theta_0-\frac{z}{1+\sigma}\Theta_1\right]=0,$$

$$\frac{\partial^2}{\partial y^2}\left[\nabla^2\chi'-\frac{1}{1+\sigma}.\Theta_0-\frac{z}{1+\sigma}\Theta_1\right]=0,$$

$$\frac{\partial^2}{\partial x\partial y}\left[\nabla^2\chi'-\frac{1}{1+\sigma}\Theta_0-\frac{z}{1+\sigma}\Theta_1\right]=0.$$

These equations show that

$$\nabla^2\chi'-\frac{1}{1+\sigma}\Theta_0-\frac{z}{1+\sigma}\Theta_1$$

is a function which is linear in x, y,

$$f_1''(z)+xf_2''(z)+yf_3''(z),$$

say. This function contributes to χ' a part given by

$$f_1(z)+xf_2(z)+yf_3(z)$$

which can obviously be omitted without altering the stresses X_x, Y_y, X_y. Hence we can write

$$\nabla^2\chi'=\frac{1}{1+\sigma}\Theta_0+\frac{z}{1+\sigma}\Theta_1. \tag{18}$$

Since $Z_z=0$, we have from (5), (17a) and (17b),

$$\Theta=X_x+Y_y=\nabla_1{}^2\chi'+\frac{2z}{1+\sigma}\Theta_1+\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2\psi}{\partial z^2}.$$

The comparison of this equation with the equation (8) shows that

$$\nabla_1{}^2\chi'=\Theta_0-\frac{1-\sigma}{1+\sigma}z\Theta_1-\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2\psi}{\partial z^2}. \tag{19}$$

Subtracting the equation (19) from the equation (18), we get

$$\frac{\partial^2\chi'}{\partial z^2}=-\frac{\sigma}{1+\sigma}\Theta_0+\frac{2-\sigma}{1+\sigma}z\Theta_1+\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2\psi}{\partial z^2}. \tag{20}$$

Integrating this equation twice with respect to z, we have

$$\chi'=\chi_0+z\chi_1-\frac{1}{2}\cdot\frac{\sigma}{1+\sigma}z^2\Theta_0+\frac{1}{6}\cdot\frac{2-\sigma}{1+\sigma}z^3\Theta_1+\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right], \tag{21}$$

where χ_0, χ_1 are functions of x, y only. Substituting from (21) in (19) and remembering that Θ_0, Θ_1 are plane harmonic functions in x, y, and ψ is a harmonic function, we get the equations satisfied by χ_0 and χ_1. They are

$$\nabla_1{}^2\chi_0=\Theta_0, \quad \nabla_1{}^2\chi_1=-\frac{1-\sigma}{1+\sigma}\Theta_1. \tag{22}$$

The function ψ appears implicitly in X_x, Y_y, X_y through the function χ', and also explicitly in X_x, Y_y as shown by the equations (17a) and (17b). Separating the terms

containing ψ, we find that ψ contributes to X_x, Y_y, X_y, the parts

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right], \quad \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right], \quad -\frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right]$$

respectively, which reduce to

$$-2\frac{\partial^4\psi}{\partial x^2\partial y^2}, \quad 2\frac{\partial^4\psi}{\partial x^2\partial y^2}, \quad -\frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right]$$

respectively, if we take account of the equation (16).

Thus we find that if we assume $Z_z = 0$ identically, the five other stress components which satisfy the equations (1) and (4) are given by

$$X_x = \frac{\partial^2\chi}{\partial y^2}+\frac{z}{1+\sigma}\Theta_1-2\frac{\partial^4\psi}{\partial x^2\partial y^2}, \tag{23a}$$

$$Y_y = \frac{\partial^2\chi}{\partial x^2}+\frac{z}{1+\sigma}\Theta_1+2\frac{\partial^4\psi}{\partial x^2\partial y^2}, \tag{23b}$$

$$X_y = -\frac{\partial^2\chi}{\partial x\partial y}-\frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right], \tag{23c}$$

$$X_z = \frac{h^2-z^2}{2(1+\sigma)}\cdot\frac{\partial\Theta_1}{\partial x}-\frac{\partial^4\psi}{\partial x\partial y^2\partial z}, \tag{23d}$$

$$Y_z = \frac{h^2-z^2}{2(1+\sigma)}\cdot\frac{\partial\Theta_1}{\partial y}+\frac{\partial^4\psi}{\partial x^2\partial y\partial z}, \tag{23e}$$

where

$$\chi = \chi_0+z\chi_1-\frac{1}{2}\cdot\frac{\sigma}{1+\sigma}z^2\Theta_0+\frac{1}{6}\cdot\frac{2-\sigma}{1+\sigma}z^3\Theta_1. \tag{24}$$

The functions Θ_0, Θ_1, ψ satisfy the equations

$$\nabla_1^2\Theta_0 = 0, \quad \nabla_1^2\Theta_1 = 0, \quad \nabla^2\psi = 0, \tag{25}$$

and the functions χ_0, χ_1 satisfy the equations

$$\nabla_1^2\chi_0 = \Theta_0, \quad \nabla_1^2\chi_1 = -\frac{1-\sigma}{1+\sigma}\Theta_1. \tag{26}$$

The expressions for the five stress components given in the equations (23) are very convenient for taking into consideration the boundary conditions on the plane faces of the plate.

Determination of the function ψ

Let us now introduce the boundary conditions (2) in the equations (23d) and (23e). We get

$$\frac{\partial^4\psi}{\partial x\partial y^2\partial z} = 0, \quad \frac{\partial^4\psi}{\partial x^2\partial y\partial z} = 0, \tag{27}$$

on the faces $z = \pm h$,

Let us seek solutions of the equation (16) which satisfy the boundary conditions (27). Expanding ψ in powers of z, we write

$$\psi = \sum_{n=0}^{\infty} \psi_n z^n,$$

where ψ_n is a function of x, y only. Then

$$\nabla^2\psi = \sum_{n=0}^{\infty} [z^n \nabla_1^2 \psi_n + n(n-1) z^{n-2} \psi_n] = 0.$$

Equating the coefficient of z^n in this equation to zero, we get

$$\nabla_1^2 \psi_n + (n+2)(n+1)\psi_{n+2} = 0, \quad n \geqq 0. \tag{28}$$

This equation shows that the ψ's divide themselves into two groups, the odd ψ's satisfying the relations

$$\nabla_1^2[(2n-1)!\,\psi_{2n-1}] + (2n+1)!\,\psi_{2n+1} = 0, \quad n \geqq 1, \tag{29}$$

and the even ψ's satisfying the relations

$$\nabla_1^2[(2n-2)!\,\psi_{2n-2}] + (2n)!\,\psi_{2n} = 0, \quad n \geqq 1, \tag{30}$$

where in the case $n = 1$, we replace $0!$ by 1.

Let us first restrict ourselves to the equation (29) and try to satisfy it by

$$(2n+1)!\,\psi_{2n+1} = f(2n+1)\Psi, \tag{31}$$

where Ψ is a function of x, y and is independent of n. Substituting in (29) we get

$$\nabla_1^2\Psi + \frac{f(2n+1)}{f(2n-1)}\Psi = 0.$$

In order that Ψ may be independent of n, we must have

$$\frac{f(2n+1)}{f(2n-1)} = \text{constant} = \pm k^2, \text{ say.}$$

This equation shows that we can take

$$f(2n+1) = (\pm 1)^n k^{2n+1},$$

and then

$$\nabla_1^2\Psi \pm k^2\Psi = 0.$$

Taking the *plus* sign and restricting ourselves to the terms with odd powers of z in ψ, we have

$$\sum_{n=0}^{\infty} \psi_{2n+1} z^{2n+1} = \Psi \sum_{n=0}^{\infty} \frac{k^{2n+1} z^{2n+1}}{(2n+1)!} = \Psi \sinh kz, \tag{32}$$

and in this case Ψ satisfies the equation

$$\nabla_1^2\Psi + k^2\Psi = 0. \tag{33}$$

If we take the *minus* sign and restrict ourselves to terms with odd powers of z, we have

$$\sum_{n=0}^{\infty} \psi_{2n+1} z^{2n+1} = \Psi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} k^{2n+1} z^{2n+1} = \Psi \sin kz, \tag{34}$$

and in this case

$$\nabla_1^2\Psi - k^2\Psi = 0. \tag{35}$$

We could have, from the very beginning, assumed solutions of the forms (32) and (34), and then we would have obtained the equations satisfied by Ψ.

Now, taking into consideration the boundary conditions (27) on the plane faces of the plate, we see at once that we will have to reject the solution (32) and retain the solution (34). In this latter case, k must satisfy the equation

$$\cos kh = 0.$$

This gives

$$k = \frac{(2m+1)\pi}{2h}, \quad m = 0, 1, 2, \ldots$$

If the solution be not of the form (31), *i.e.*, if the equations (29) do not reduce to a single equation, we have from (29),

$$\psi_{2n+1} = \frac{(-1)^n}{(2n+1)!}\nabla_1^{2n}\psi_1. \tag{36}$$

Restricting ourselves to terms with odd powers of z in ψ, and using the boundary conditions (27), we get

$$\left[1+\sum_{n=1}^{\infty}(-1)^n \cdot \frac{h^{2n}}{(2n)!}\nabla_1^{2n}\right]\frac{\partial^3\psi_1}{\partial x\partial y^2} = 0, \tag{37}$$

$$\left[1+\sum_{n=1}^{\infty}(-1)^n \cdot \frac{h^{2n}}{(2n)!}\nabla_1^{2n}\right]\frac{\partial^3\psi_1}{\partial x^2\partial y} = 0. \tag{38}$$

The series

$$1+\sum_{n=1}^{\infty}(-1)^n \cdot \frac{h^{2n}}{(2n)!}$$

is absolutely convergent for all values of h and its sum is $\cos h$. Then if

$$1+\sum_{n=1}^{\infty}c_{2n}h^{2n} = \left[1+\sum_{n=1}^{\infty}(-1)^n\frac{h^{2n}}{(2n)!}\right]^{-1}$$

be its reciprocal series, it is absolutely convergent when $|h| < \pi/2$. By choosing the unit of length suitably, we can always make h satisfy this condition. If we therefore operate on both sides of the equations (37) and (38) by the operator.

$$1+\sum_{n=1}^{\infty}c_{2n}h^{2n}\nabla_1^{2n}$$

we get

$$\frac{\partial^3\psi_1}{\partial x\partial y^2} = 0, \quad \frac{\partial^3\psi_1}{\partial x^2\partial y} = 0. \tag{39}$$

The equation (36) then gives $\quad \dfrac{\partial^3\psi_{2n+1}}{\partial x\partial y^2} = 0, \quad \dfrac{\partial^3\psi_{2n+1}}{\partial x^2\partial y} = 0. \qquad (40)$

It is seen at once from (23) that ψ_{2n+1} does not contribute to the stresses. Therefore without any loss of generality we can take $\psi_{2n+1} = 0$.

Hence if we restrict ourselves to terms with odd powers of z in ψ, we find on using the boundary conditions (27) that

$$\psi = \sum_{n=1}^{\infty} \Psi_{2n+1} \sin \frac{(2n+1)\pi z}{2h},$$

where Ψ_{2n+1} satisfies the equation

$$\nabla_1^2 \Psi_{2n+1} - \frac{(2n+1)^2\pi^2}{4h^2} \Psi_{2n+1} = 0.$$

Proceeding exactly in a similar manner with the terms with even powers of z in ψ, we find that if the equations (30) reduce to a single equation,

$$\psi = \Psi_{2n} \cos \frac{n\pi z}{h}$$

satisfies the equation (16) and the boundary conditions (27), provided

$$\nabla_1^2 \Psi_{2n} - \frac{n^2\pi^2}{h^2} \Psi_{2n} = 0.$$

If the equations (30) do not reduce to a single equation, we have proceeding as before,

$$\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{h^{2n}}{(2n+1)!} \nabla_1^{2n}\right] \frac{\partial^3}{\partial x \partial y^2} \nabla_1^2 \psi_0 = 0, \tag{41}$$

$$\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{h^{2n}}{(2n+1)!} \nabla_1^{2n}\right] \frac{\partial^3}{\partial x^2 \partial y} \nabla_1^2 \psi_0 = 0. \tag{42}$$

We see at once that the series

$$1 + \sum_{n=1}^{\infty} d_{2n} h^{2n} = \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} h^{2n}\right]^{-1}$$

is absolutely convergent when $|h| < \pi$. Hence operating on both sides of (41) and (42) by the operator

$$1 + \sum_{n=1}^{\infty} d_{2n} h^{2n} \nabla_1^{2n},$$

we get

$$\nabla_1^2\left(\frac{\partial^3 \psi_0}{\partial x \partial y^2}\right) = 0, \qquad \nabla_1^2\left(\frac{\partial^3 \psi_0}{\partial x^2 \partial y}\right) = 0$$

With the help of the equations (30), we find that

$$\frac{\partial^3 \psi_{2n}}{\partial x \partial y^2} = 0, \qquad \frac{\partial^3 \psi_{2n}}{\partial x^2 \partial y} = 0, \qquad n \geqq 1.$$

Thus the terms $\psi_{2n} z^{2n}$ $(n \geqq 1)$ in ψ do not contribute to the stresses. Also, as ψ_0 is independent of z, it does not contribute to X_z, Y_z and can therefore be omitted.

Combining the two solutions we have obtained, we see that the solution of the equation (16) which satisfies the boundary conditions (27) is given by

$$\psi = \sum_{n=0}^{\infty} \Psi_{2n} \cos \frac{n\pi z}{h} + \sum_{n=0}^{\infty} \Psi_{2n+1} \sin \frac{(2n+1)\pi z}{2h}, \tag{43}$$

where

$$\nabla_1^2 \Psi_m - \frac{m^2\pi^2}{4h^2} \Psi_m = 0, \quad m = 0, 1, 2, \dots . \tag{44}$$

TRACTIONS ON THE CYLINDRICAL EDGE

Let us now calculate the tractions on the cylindrical edge of the plate. Let l, m, 0 be the direction cosines of the out-ward drawn normal to the edge-line. Calculating the tractions from (3) and (23), we find

$$X_\nu = \left(l\frac{\partial^2\chi}{\partial y^2} - m\frac{\partial^2\chi}{\partial x\partial y}\right) - 2l\frac{\partial^4\psi}{\partial x^2\partial y^2} - m\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) + \frac{l}{1+\sigma}z\Theta_1,$$

$$Y_\nu = -\left(l\frac{\partial^2\chi}{\partial x\partial y} - m\frac{\partial^2\chi}{\partial x^2}\right) - l\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) + 2m\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{m}{1+\sigma}z\Theta_1,$$

$$Z_\nu = \frac{h^2-z^2}{2(1+\sigma)}\left(l\frac{\partial\Theta_1}{\partial x} + m\frac{\partial\Theta_1}{\partial y}\right) - \left(l\frac{\partial^4\psi}{\partial x\partial y^2\partial z} - m\frac{\partial^4\psi}{\partial x^2\partial y\partial z}\right).$$

Using the equation (16), we write

$$X_\nu = \left(l\frac{\partial^2\chi}{\partial y^2} - m\frac{\partial^2\chi}{\partial x\partial y}\right) - 2\left(l\frac{\partial^4\psi}{\partial x^2\partial y^2} - m\frac{\partial^4\psi}{\partial x^3\partial y}\right) + m\frac{\partial^4\psi}{\partial x\partial y\partial z^2} + \frac{l}{1+\sigma}z\Theta_1,$$

$$Y_\nu = -\left(l\frac{\partial^2\chi}{\partial x\partial y} - m\frac{\partial^2\chi}{\partial x^2}\right) - 2\left(l\frac{\partial^4\psi}{\partial x\partial y^3} - m\frac{\partial^4\psi}{\partial x^2\partial y^2}\right) - l\frac{\partial^4\psi}{\partial x\partial y\partial z^2} + \frac{m}{1+\sigma}z\Theta_1,$$

$$Z_\nu = \frac{h^2-z^2}{2(1+\sigma)}\left(l\frac{\partial\Theta_1}{\partial x} + m\frac{\partial\Theta_1}{\partial y}\right) - \left(l\frac{\partial^4\psi}{\partial x\partial y^2\partial z} - m\frac{\partial^4\psi}{\partial x^2\partial y\partial z}\right).$$

Let us measure the arc s of the edge-line in such a sense that the rotation from the tangent to the edge-line in the direction s increasing to the inward-drawn normal to it has the same sense as that of the rotation from ox to oy. Then

$$l = \frac{dy}{ds}, \quad m = -\frac{dx}{ds},$$

and therefore

$$l\frac{\partial}{\partial y} - m\frac{\partial}{\partial x} = \frac{\partial}{\partial s}.$$

Let Φ_1 be the complex conjugate of Θ_1, so that

$$\frac{\partial\Theta_1}{\partial x} = \frac{\partial\Phi_1}{\partial y}, \quad \frac{\partial\Theta_1}{\partial y} = -\frac{\partial\Phi_1}{\partial x}.$$

Then we have

$$l\frac{\partial\Theta_1}{\partial x} + m\frac{\partial\Theta_1}{\partial y} = l\frac{\partial\Phi_1}{\partial y} - m\frac{\partial\Phi_1}{\partial x} = \frac{\partial\Phi_1}{ds}.$$

The tractions on the cylindrical edge are therefore,

$$X_\nu = \frac{\partial}{\partial s}\left[\frac{\partial\chi}{\partial y} - 2\frac{\partial^3\psi}{\partial x^2\partial y}\right] + m\frac{\partial^4\psi}{\partial x\partial y\partial z^2} + \frac{l}{1+\sigma}z\Theta_1, \tag{45a}$$

$$Y_\nu = -\frac{\partial}{\partial s}\left[\frac{\partial\chi}{\partial x} + 2\frac{\partial^3\psi}{\partial x\partial y^2}\right] - l\frac{\partial^4\psi}{\partial x\partial y\partial z^2} + \frac{m}{1+\sigma}z\Theta_1, \tag{45b}$$

$$Z_\nu = -\frac{\partial}{\partial s}\left[\frac{\partial^3\psi}{\partial x\partial y\partial z} - \frac{h^2-z^2}{2(1+\sigma)}\Phi_1\right]. \tag{45c}$$

Substituting from (43) in (45), we get

$$X_\nu = \frac{\partial}{\partial s}\left[\frac{\partial\chi}{\partial y} - 2\sum_{n=0}^{\infty}\frac{\partial^3\Psi_{2n}}{\partial x^2\partial y}\cos\frac{n\pi z}{h} - 2\sum_{n=0}^{\infty}\frac{\partial^3\Psi_{2n+1}}{\partial x^2\partial y}\sin\frac{(2n+1)\pi z}{2h}\right]$$

$$-\frac{m\pi^2}{h^2}\left[\sum_{n=0}^{\infty}n^2\frac{\partial^2\Psi_{2n}}{\partial x\partial y}\cos\frac{n\pi z}{h} + \frac{1}{4}\sum_{n=0}^{\infty}(2n+1)^2\frac{\partial^2\Psi_{2n+1}}{\partial x\partial y}\sin\frac{(2n+1)\pi z}{2h}\right] + \frac{l}{1+\sigma}z\Theta_1, \quad (46a)$$

$$Y_\nu = -\frac{\partial}{\partial s}\left[\frac{\partial\chi}{\partial x} + 2\sum_{n=0}^{\infty}\frac{\partial^3\Psi_{2n}}{\partial x\partial y^2}\cos\frac{n\pi z}{h} + 2\sum_{n=0}^{\infty}\frac{\partial^3\Psi_{2n+1}}{\partial x\partial y^2}\sin\frac{(2n+1)\pi z}{2h}\right]$$

$$+\frac{l\pi^2}{h^2}\left[\sum_{n=0}^{\infty}n^2\frac{\partial^2\Psi_{2n}}{\partial x\partial y}\cos\frac{n\pi z}{h} + \frac{1}{4}\sum_{n=0}^{\infty}(2n+1)^2\frac{\partial^2\Psi_{2n+1}}{\partial x\partial y}\sin\frac{(2n+1)\pi z}{2h}\right] + \frac{m}{1+\sigma}z\Theta_1, \quad (46b)$$

$$Z_\nu = \frac{\pi}{h}\frac{\partial}{\partial s}\left[\sum_{n=0}^{\infty}n\frac{\partial^2\Psi_{2n}}{\partial x\partial y}\sin\frac{n\pi z}{h} - \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{\partial^2\Psi_{2n+1}}{\partial x\partial y}\cos\frac{(2n+1)\pi z}{2h}\right] + \frac{h^2-z^2}{2(1+\sigma)}\frac{\partial\Phi_1}{\partial s}. \quad (46c)$$

Discussion of the results

We have obtained an exact solution of the equations (1) and (4) which satisfies the boundary conditions (2), and in which Z_z is identically zero. This solution is given by the equations (23), (24), (25) and (26). It proves the possibility of a state of generalized plane stress in a plate of an isotropic material whose plane faces are free from tractions. The equations (46) give the most general values of the tractions which are to be applied to the cylindrical edge of the plate in order to maintain a state of generalized plane stress in it.

The state of plane stress in the plate is obtained from the state of generalized plane stress by putting $\Psi_{2n} = 0$, $\Psi_{2n+1} = 0$, and $\Theta_1 = \beta$, where β is a constant. This shows that the assumption of a state of generalized plane stress instead of one of plane stress increases the generality of the solution by introducing a harmonic function ψ (or what is the same, the functions Ψ_m satisfying (44)) and a plane harmonic function Θ_1.

The stress system depends on the functions χ_0, χ_1, Θ_0, Θ_1 and ψ. The function χ_0 is an unrestricted plane biharmonic function in x, y, both in the state of plane stress and in the state of generalized plane stress. This function contributes to X_ν, Y_ν but not to Z_ν. Therefore, as far as terms independent of z are concerned, χ_0 places no restriction on X_ν, Y_ν, but gives $Z_\nu = 0$; only as χ_0 determines Θ_0 suitable terms of the order z^2 are to be associated with X_ν, Y_ν.

The function χ_1 is an unrestricted plane biharmonic function in x, y, in the state of generalized plane stress, but in the state of plane stress its plane harmonic part is only unrestricted while the other part is of the form $\frac{1}{4}\beta(x^2+y^2)$. As χ_1 determines Θ_1, suitable terms of the orders z and z^3 are to be associated with X_ν, Y_ν and suitable terms independent of z and of the order z^2 are to be associated with Z_ν.

Thus, as far as terms corresponding to χ_0, Θ_0 are concerned, no greater generality is introduced in our solution by assuming a state of generalized plane stress than is present in a state of plane stress. These terms correspond to the stretching of the plate by forces

in its plane. As far as terms corresponding to χ_1 and Θ_1 are concerned, the solution for generalized plane stress is certainly more general than that for plane stress. But as these terms determine Θ_1 and therefore Φ_1, they introduce appropriate tractions Z_ν. The terms in χ_1, Θ_1 correspond to the bending of the plate.

The function ψ is a harmonic function given by the equations (43) and (44). It introduces terms in $\sin(n\pi z/h)$ and $\cos\{(2n+1)\pi z/(2h)\}$ in Z_ν, and terms in $\cos(n\pi z/h)$ and $\sin\{(2n+1)\pi z/(2h)\}$ in X_ν, Y_ν. These terms slightly increase the generality of our solution in that they permit variations in X_ν, Y_ν, Z_ν with z. But when we prescribe the terms in Z_ν corresponding to $\sin(n\pi z/h)$ and $\cos\{(2n+1)\pi z/(2h)\}$, the terms in X_ν, Y_ν depending on $\cos(n\pi z/h)$ and $\sin\{(2n+1)\pi z/(2h)\}$ are no longer at our disposal.

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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1945 to the Annual General Meeting of the Calcutta Mathematical Society

The Council of the Calcutta Mathematical Society have the pleasure to submit the following report on the general concerns of the Society for the year 1945, as required by the provisions of Rule 25.

The Council:—The Council of the Society for the year 1945 consisting of the officers and members elected at the last Annual General Meeting together with the Editorial Secretary, was constituted as follows:—

President

Prof. N. R. Sen

Vice-Presidents

Prof. F. W. Levi, Prof. M. R. Siddiqi,
Dr. C. N. Srinivasienger, Prof. A. C. Banerjee,
Dr. R. N. Sen

Treasurer

Mr. S. C. Ghosh

Secretary

Dr. S. K. Chakrabarty (upto March, 1945)
Mr. U. R. Burman (from April, 1945)

Editorial Secretary

Mr. S. Gupta

Other Members of the Council

Mr. B. M. Sen,	Prof. V. V. Narlikar	Mr. B. B. Sen
Prof. M. N. Saha	Mr. N. N. Ghosh	Dr. S. Ghosh
Dr. S. S. Pillai	Prof. C. V. H. Rao	Mr. B. C. Chatterjee.
Dr. S. R. Sen Gupta	Prof. N. M. Basu	Mr. P. K. Ghosh (*Assistant Secretary*)

The Council have been unfortunate in losing the services in the early part of the year of its Secretary Dr. S. K. Chakrabarty whose appointment in the Indian Meteorological Service has necessitated his stay outside Calcutta. During the long period of his association with the Society as its Secretary, Dr. Chakrabarty has rendered the most generous services, which the council deeply appreciate with grateful thanks.

General:—The various activities of the Society have been carried on throughout the year in accordance with the usual wartime arrangements. With the prospect of early return to normal conditions, the Society can look forward to the gradual revival of the normal activities, specially the holding of symposiums from time to time on various branches of Mathematics.

Membership:—The council record with regret the death of the following two members, their deaths occuring in 1944.

*Prof. D. E. Smith

Mr. P. N. Tandon

During the year under review 10 new members were elected.

Meetings during 1945:—The Council held five meetings during the year, and there were six Ordinary General Meetings which were devoted to the reading of original papers communicated to the Society.

Publications:—Four numbers of the *Bulletin* were published during the year 1945, *viz.*, No. 4 of Vol. 36 and Nos. 1, 2, 3 of Vol. 37, so that at the close of the year the Bulletin was in arrear by one number only.

During the year, the paper position has slightly improved and the cut made in the number of reprints allowed to authors has been restored.

The Society received with grateful thanks a grant of Rs. 250 from the Rockefeller Foundation Grant, in 1944, for the improvement of the Bulletin. The Council report with pleasure that this amount has been utilised partly in the purchase of paper and partly in procuring certain essential mathematical types, which have largely contributed towards the improvement of the Journal.

The authorities of the Calcutta University have continued to lend their usual magnanimous services in printing the Bulletin free of charge, and the officers and members of the staff of the University Press have given their every sympathetic and active co-operation in bringing out the Bulletin regularly in these difficult times. The council take this opportunity of offering them its very sincere thanks.

Exchanges of Publications:—The distribution of the Bulletin to countries with which communication was impracticable during the war period is being gradually restored. While it is hoped shortly to resume the despatch of current publications to these countries, it will not be immediately possible to make up the war-time gaps for all the Institutions on the exchange list, though one or two complete sets of wartime publications are being sent out as channels become available.

The Library:—The use of the Library continues to increase, and during the year under review 170 books were borrowed as compared with the figures of 160 books in the previous year. A general improvement in working of the Library will also be noticed, thanks to the efforts of the Assistant Secretary, Mr. P. K. Ghosh.

* Honorary Member

Finance :—The Annual Accounts from January 1, to December 31, 1945, have been presented to the Council in the usual form by the auditors, Dr. B. S. Ray and Mr. A. C. Choudhury, who deserve the Council's grateful thanks for their honorary services.

It has not been possible this year also to settle up the foreign accounts due to obvious difficulties. The Council, however, hopes that this state of affairs will not continue long, as greater facilities for dealing with foreign exchange are now gradually being available. The financial position of the Society as a whole remains much the same as in the past few years.

The Society's Office and Staff :—Wartime difficulties still existing, it has not been possible to secure a good assistant for the office with the remuneration which the Society can now offer. The work is, however, being carried on with the help of a part-time assistant, who is working on a small allowance.

The Council regret to report that the bearer of the Society, a thoroughly dependable and trustworthy old man, who has been rendering very useful services for so many years, has been compelled to take long leave on grounds of health. It will be a matter of satisfaction to see him fit again and resume the work.

CALCUTTA MATHEMATICAL SOCIETY

Receipts and Disbursements Accounts of the Calcutta Mathematical Society for the year ending 31st December, 1945

Receipts

				Rs. as. p.
1.	Opening balance:—			
	(a) Cash With Secretary ...		5 5 0	
	(b) Balance at Bank			
	(i) Imperial Bank of India	1083 2 6		
	(ii) Do. (K. K. G. P. Fund)	146 13 3		
	(iii) Bengal Central Bank	653 0 0		
	(iv) Do. realised from the Suspense Account of the previous year) ...	12 13 0		
	(v) P. O. Savings Bank	561 5 3		
			2457 2 0	
	(c) G. P. Notes (General Fund) (Face value Rs. 6 000) ...		5663 11 6	
	(d) G. P. Notes (K. K. G. P Fund) (Face Value Rs. 2,000) ...		1987 7 9	
				10063 10 3
2.	Membership Subscription ...		708 2 0	
3.	Admission Fee ...		40 0 0	
4.	Sale Proceeds ...		280 12 0	
				1118 14 0
5	Interest—			
	(a) P. O. Savings Bank ...		8 7 0	
	(b) G. P. Notes (General Fund) ...		210 0 0	
	(c) Do. (K. K. G. P. Fund) ...		70 0 0	
				288 7 0
	Total ..			11,470 15 3

Disbursements

				Rs. as. p.
1.	Establishment ...		597 2 0	
2.	Meetings ...		76 6 6	
3.	Books and Journals (including charges for Book binding) ...		93 12 0	
4.	Bulletins			
	(a) Papers, Blocks and Types	548 5 9		
	(b) Postage	43 5 9		
			591 11 6	
5.	Printing and Stationary ..		57 13 9	
6.	Postage ...		68 6 0	
7	Bank Charges ...		12 2 0	
8	Miscellaneous (including conveyance charges) ...		28 7 0	
				1,525 12 9
	Closing Balance			
	(a) Cash with Secretary .		7 1 3	
	(b) Balance at Bank			
	(i) Imperial Bank of India	1292 2 6		
	(ii) Do. (K K. G. P. Fund)	216 5 3		
	(iii) Bengal Central Bank	258 10 0		
	(iv) P O. Savings Bank	569 12 3		
			2336 14 0	
	(c) G. P. Notes (General Fund (Face Value Rs. 6,000) ...		5663 11 6	
	(d) G. P. Notes (K K. G P. Fund) (Face value Rs 2,000) ...		1987 7 9	
				9,945 2 6
	Total .			11,470 15 3

To
The Members of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto, and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

B. S. RAY
A. C. CHOUDHURY
Auditors

ON GENERALIZED PLANE STRESS IN AN AEOLOTROPIC PLATE

By

S. Ghosh

(*Received March 8, 1946*)

Introduction

As an exact solution of the problem of equilibrium of a plate of an isotropic material stretched and bent by forces applied to its rim is very difficult to determine, it is customary for the simplification of the problem to assume a state of plane stress (Love, 1927, p. 137) or a state of generalized plane stress (Coker and Filon, 1931, p. 126) in the plate. On this assumption, the average stresses and displacements in the plate can be very easily determined. It is well-known (Love, 1927, p. 206) that the assumption of a state of plane stress leads to single-valued displacements in the plate, showing thereby that such a state of stress is possible in an isotropic plate. It has been shown (Ghosh, 1946, p. 45) on the assumption of a state of generalized plane stress in the plate that an exact solution of the equations of equilibrium can be obtained which satisfies the compatibility equations for the stresses as well as the boundary conditions on the plane faces of the plate. The existence of this solution proves the possibility of a state of generalized plane stress in an isotropic plate.

In a recent paper (Ghosh, 1942, p. 157) attention has been called to the fact that the adoption of such a procedure for the determination of average stresses in a plate of an aeolotropic material is fraught with grave risks, unless we can prove the possibility of the state of stress we assume. In fact it has been shown there that a state of plane stress is possible in a plate of an aeolotropic material with three perpendicular planes of symmetry, only if the stresses are derived from a stress function which is a polynomial of the sixth degree in x, y ($z = 0$ being the middle plane of the plate). This shows that the determination of average stresses cannot be made on the assumption of a state of plane stress in the plate when the stress resultants and stress couples are arbitrarily prescribed on its rim. Green (1945, p. 224) is of opinion that this argument does not necessarily apply to generalized plane stress, for the restrictions involved in this assumption are less stringent than those required by the theory of plane stress. A wider generality is no doubt possessed by the stress distribution in a state of generalized plane stress, but the question is whether this generality is sufficient to take account of the arbitrarily prescribed boundary conditions on the rim of the plate.

The object of the present paper is to examine the possibility of a state of generalized plane stress in a plate of an aeolotropic material with three perpendicular planes of symmetry, the plane faces of the plate being parallel to one of these planes of symmetry. The components of stress have been expressed in terms of two stress functions χ and ψ in equations (9), and with the help of the stress-strain relations for the aeolotropic

material and the equations of compatibility for strain, relations between χ and ψ have been established in the equations (12). From these equations it is deduced that ψ satisfies two homogeneous linear partial differential equations of the sixth order which are not, in general, compatible unless ψ is a polynomial of degree not higher than 13 in x, y, z. As these stress functions cannot lead to an arbitrary distribution of stress resultants and stress couples on the rim of the plate, a state of generalized plane stress is not in general possible in such a plate.

Fundamental equations

Let us consider a plate of an aeolotropic material with three perpendicular planes of symmetry, and let the plane faces of the plate be parallel to one of these planes of symmetry. Let us suppose that the plate is in equilibrium under the action of forces acting on its cylindrical edge. Let the middle plane of the plate be taken as the plane $z = 0$. The stress equations of equibrium of the plate are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z} = 0, \tag{1a}$$

$$\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z} = 0, \tag{1b}$$

$$\frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z} = 0. \tag{1c}$$

If $2h$ be the thickness of the plate, the conditions that the plane faces of the plate are free from tractions are

$$X_z = Y_z = Z_z = 0, \tag{2}$$

when $z = \pm h$.

The stress-strain relations for the aeolotropic material are (Ghosh, 1942, p. 163)

$$e_{xx} = s_{11}X_x + s_{12}Y_y + s_{13}Z_z, \qquad e_{yz} = s_{44}Y_z, \tag{3a}$$

$$e_{yy} = s_{21}X_x + s_{22}Y_y + s_{23}Z_z, \qquad e_{zx} = s_{55}Z_x, \tag{3b}$$

$$e_{zz} = s_{31}X_x + s_{32}Y_y + s_{33}Z_z, \qquad e_{xy} = s_{66}X_y. \tag{3c}$$

The components for strain satisfy the compatibility relations,

$$\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{yz}}{\partial y \partial z}, \qquad 2\frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right), \tag{4a}$$

$$\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2} = \frac{\partial^2 e_{zx}}{\partial z \partial x}, \qquad 2\frac{\partial^2 e_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right), \tag{4b}$$

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}, \qquad 2\frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right). \tag{4c}$$

STRESS FUNCTIONS

In a state of generalized plane stress,

$$Z_z = 0, \tag{5}$$

while X_z and Y_z are not identically zero. Substituting from (5) in (1c), we see that we can express X_z, Y_z in terms of a single function ϕ by means of the formulae

$$X_z = -\frac{\partial\phi}{\partial y}, \quad Y_z = \frac{\partial\phi}{\partial x}. \tag{6}$$

Taking

$$\phi = \frac{\partial^3\psi}{\partial x\partial y\partial z}, \tag{7}$$

and substituting from (6) and (7) in (1a) and (1b), we have

$$\frac{\partial}{\partial x}\left(X_x - \frac{\partial^4\psi}{\partial y^2\partial z^2}\right) + \frac{\partial X_y}{\partial y} = 0, \tag{8a}$$

$$\frac{\partial X_y}{\partial x} + \frac{\partial}{\partial y}\left(Y_y + \frac{\partial^4\psi}{\partial x^2\partial z^2}\right) = 0. \tag{8b}$$

These equations show that a function χ exists such that

$$X_x = \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^4\psi}{\partial y^2\partial z^2}, \tag{9a}$$

$$Y_y = \frac{\partial^2\chi}{\partial x^2} - \frac{\partial^4\psi}{\partial x^2\partial z^2}, \tag{9b}$$

$$X_y = -\frac{\partial^2\chi}{\partial x\partial y}. \tag{9c}$$

To these we add the equations

$$X_z = -\frac{\partial^4\psi}{\partial x\partial y^2\partial z}, \quad Y_z = \frac{\partial^4\psi}{\partial x^2\partial y\partial z}. \tag{9d}$$

We observe in this connection that ψ appears in the equations (8a) and (8b) through the stress components X_z, Y_z, so that any value of ψ which makes $X_z = Y_z = 0$ can be omitted without altering the other stress components.

Calculating the strain components from the equations (3), (5) and (9), we get

$$e_{xx} = \left(s_{11}\frac{\partial^2\chi}{\partial y^2} + s_{12}\frac{\partial^2\chi}{\partial x^2}\right) + \frac{\partial^2}{\partial z^2}\left(s_{11}\frac{\partial^2\psi}{\partial y^2} - s_{12}\frac{\partial^2\psi}{\partial x^2}\right), \tag{10a}$$

$$e_{yy} = \left(s_{12}\frac{\partial^2\chi}{\partial y^2} + s_{22}\frac{\partial^2\chi}{\partial x^2}\right) + \frac{\partial^2}{\partial z^2}\left(s_{12}\frac{\partial^2\psi}{\partial y^2} - s_{22}\frac{\partial^2\psi}{\partial x^2}\right), \tag{10b}$$

$$e_{zz} = \left(s_{13}\frac{\partial^2\chi}{\partial y^2} + s_{23}\frac{\partial^2\chi}{\partial x^2}\right) + \frac{\partial^2}{\partial z^2}\left(s_{13}\frac{\partial^2\psi}{\partial y^2} - s_{23}\frac{\partial^2\psi}{\partial x^2}\right), \tag{10c}$$

$$e_{yz} = s_{44}\frac{\partial^4\psi}{\partial x^2\partial y\partial z}, \quad e_{zx} = -s_{55}\frac{\partial^4\psi}{\partial x\partial y^2\partial z}, \quad e_{xy} = -s_{66}\frac{\partial^2\chi}{\partial x\partial y}. \tag{10d}$$

Substituting from the equations (10) in the equations (4), we find that the functions χ and ψ satisfy the six equations

$$s_{22}\frac{\partial^4\chi}{\partial x^2\partial z^2}+\frac{\partial^2}{\partial y^2}\left[s_{23}\frac{\partial^2\chi}{\partial x^2}+s_{13}\frac{\partial^2\chi}{\partial y^2}+s_{12}\frac{\partial^2\chi}{\partial z^2}\right]$$
$$=\frac{\partial^4}{\partial z^4}\left[s_{22}\frac{\partial^2\psi}{\partial x^2}-s_{12}\frac{\partial^2\psi}{\partial y^2}\right]+\frac{\partial^4}{\partial y^2\partial z^2}\left[(s_{44}+s_{23})\frac{\partial^2\psi}{\partial x^2}-s_{13}\frac{\partial^2\psi}{\partial y^2}\right], \tag{11a}$$

$$s_{11}\frac{\partial^4\chi}{\partial y^2\partial z^2}+\frac{\partial^2}{\partial x^2}\left[s_{23}\frac{\partial^2\chi}{\partial x^2}+s_{13}\frac{\partial^2\chi}{\partial y^2}+s_{12}\frac{\partial^2\chi}{\partial z^2}\right]$$
$$=\frac{\partial^4}{\partial z^4}\left[s_{12}\frac{\partial^2\psi}{\partial x^2}-s_{11}\frac{\partial^2\psi}{\partial y^2}\right]-\frac{\partial^4}{\partial x^2\partial z^2}\left[(s_{55}+s_{13})\frac{\partial^2\psi}{\partial y^2}-s_{23}\frac{\partial^2\psi}{\partial x^2}\right], \tag{11b}$$

$$s_{22}\frac{\partial^4\chi}{\partial x^4}+(s_{66}+2s_{12})\frac{\partial^4\chi}{\partial x^2\partial y^2}+s_{11}\frac{\partial^4\chi}{\partial y^4}=\frac{\partial^2}{\partial z^2}\left[s_{22}\frac{\partial^4\psi}{\partial x^4}-s_{11}\frac{\partial^4\psi}{\partial y^4}\right] \tag{11c}$$

$$\frac{\partial^2}{\partial y\partial z}\left[2s_{11}\frac{\partial^2\chi}{\partial y^2}+(s_{66}+2s_{12})\frac{\partial^2\chi}{\partial x^2}\right]$$
$$=-2\frac{\partial^4}{\partial y\partial z^3}\left[s_{11}\frac{\partial^2\psi}{\partial y^2}-s_{12}\frac{\partial^2\psi}{\partial x^2}\right]-\frac{\partial^4}{\partial x^2\partial y\partial z}\left[s_{44}\frac{\partial^2\psi}{\partial x^2}+s_{55}\frac{\partial^2\psi}{\partial y^2}\right], \tag{11d}$$

$$\frac{\partial^2}{\partial x\partial z}\left[2s_{23}\frac{\partial^2\chi}{\partial x^2}+(s_{66}+2s_{12})\frac{\partial^2\chi}{\partial y^2}\right]$$
$$=-2\frac{\partial^4}{\partial x\partial z^3}\left[s_{12}\frac{\partial^2\psi}{\partial y^2}-s_{22}\frac{\partial^2\psi}{\partial x^2}\right]+\frac{\partial^4}{\partial x\partial y^2\partial z}\left[s_{44}\frac{\partial^2\psi}{\partial x^2}+s_{55}\frac{\partial^2\psi}{\partial y^2}\right], \tag{11e}$$

$$\frac{\partial^2}{\partial x\partial y}\left[2s_{23}\frac{\partial^2\chi}{\partial x^2}+2s_{13}\frac{\partial^2\chi}{\partial y^2}-s_{66}\frac{\partial^2\chi}{\partial z^2}\right]=\frac{\partial^4}{\partial x\partial y\partial z^2}\left[(s_{44}+2s_{23})\frac{\partial^2\psi}{\partial x^2}-(s_{55}+2s_{13})\frac{\partial^2\psi}{\partial y^2}\right]. \tag{11f}$$

Conditions for generalized plane stress

When $\psi=0$ we have $X_z=Y_z=0$, so that the plate is in a state of plane stress. We have already found in a previous paper (Ghosh, 1942) that only a particular distribution of surface tractions on the cylindrical edge can give rise to a state of plane stress in the plate.

The stress-system in a state of generalized plane stress consists of two parts in one of which $\psi=0$ and in the other ψ is not zero. Omitting the part of χ which corresponds to $\psi=0$ and therefore gives rise only to a particular distribution of tractions on the rim, we restrict ourselves to the part corresponding to the non-zero value of ψ. This means that in integrating the equation (11), we retain only the particular solutions of these equations. The equations (11d), (11e) and (11f) give

$$2s_{11}\frac{\partial^2\chi}{\partial y^2}+(s_{66}+2s_{12})\frac{\partial^2\chi}{\partial x^2}=-2\frac{\partial^2}{\partial z^2}\left[s_{11}\frac{\partial^2\psi}{\partial y^2}-s_{12}\frac{\partial^2\psi}{\partial x^2}\right]-\frac{\partial^2}{\partial x^2}\left[s_{44}\frac{\partial^2\psi}{\partial x^2}+s_{55}\frac{\partial^2\psi}{\partial y^2}\right], \tag{12a}$$

$$2s_{22}\frac{\partial^2\chi}{\partial x^2}+(s_{66}+2s_{12})\frac{\partial^2\chi}{\partial y^2}=-2\frac{\partial^2}{\partial z^2}\left[s_{12}\frac{\partial^2\psi}{\partial y^2}-s_{22}\frac{\partial^2\psi}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[s_{44}\frac{\partial^2\psi}{\partial x^2}+s_{55}\frac{\partial^2\psi}{\partial y^2}\right], \tag{12b}$$

$$2s_{23}\frac{\partial^2\chi}{\partial x^2}+2s_{13}\frac{\partial^2\chi}{\partial y^2}-s_{66}\frac{\partial^2\chi}{\partial z^2}=\frac{\partial^2}{\partial z^2}\left[(s_{44}+2s_{23})\frac{\partial^2\psi}{\partial x^2}-(s_{55}+2s_{13})\frac{\partial^2\psi}{\partial y^2}\right]. \tag{12c}$$

Differentiating (12b) twice with respect to z and (12c) twice with respect to y and adding we get (11a). Similarly, (12c) and (12a) lead to (11b), and (12a) and (12b) to (11c) Therefore in a state of generalized plane stress the six equations (11) reduce to the three equations (12), provided we restrict ourselves to the part of χ which arises from ψ alone. Hence for a state of generalized plane stress to be possible, the equations (12) satisfied by the stress functions χ and ψ must be compatible.

Eliminating χ between (12a) and (12b) and also between (12a) and (12c), we see that ψ satisfies the two equations

$$s_{22}s_{44}\frac{\partial^6\psi}{\partial x^6}+[s_{22}s_{55}+s_{44}(s_{66}+2s_{12})]\frac{\partial^6\psi}{\partial x^4\partial y^2}+[s_{11}s_{44}+s_{55}(s_{66}+2s_{12})]\frac{\partial^6\psi}{\partial x^2\partial y^4}+s_{11}s_{55}\frac{\partial^6\psi}{\partial y^6}$$
$$+s_{22}s_{66}\frac{\partial^6\psi}{\partial x^4\partial z^2}+[4s_{11}s_{22}-2s_{12}(s_{66}+2s_{12})]\frac{\partial^6\psi}{\partial x^2\partial y^2\partial z^2}+s_{11}s_{66}\frac{\partial^6\psi}{\partial y^4\partial z^2}=0, \quad (13a)$$

$$s_{23}s_{44}\frac{\partial^6\psi}{\partial x^6}+[s_{23}s_{55}+s_{13}s_{44}]\frac{\partial^6\psi}{\partial x^4\partial y^2}+s_{13}s_{55}\frac{\partial^6\psi}{\partial x^2\partial y^4}+[s_{12}s_{44}+s_{23}s_{66}]\frac{\partial^6\psi}{\partial x^4\partial z^2}$$
$$+[s_{11}(s_{44}+4s_{23})-s_{12}(s_{55}+4s_{13})-s_{66}(s_{55}+s_{13})]\frac{\partial^6\psi}{\partial x^2\partial y^2\partial z^2}-s_{11}s_{55}\frac{\partial^6\psi}{\partial y^4\partial z^2}$$
$$+s_{12}s_{66}\frac{\partial^6\psi}{\partial x^2\partial z^4}-s_{11}s_{66}\frac{\partial^6\psi}{\partial y^2\partial z^4}=0. \quad (13b)$$

The equations (13a) and (13b) cannot be broken up into partial differential equations of lower orders unless suitable relations exist between the elastic constants which can only happen under exceptional circumstances. These equations are homogeneous linear partial differential equations of the sixth order. If ψ be a homogeneous polynomial of degree n in x, y, z, so that it contains $\frac{1}{2}(n+1)(n+2)$ arbitrary constants, then the left-hand side of each of these equations is a homogeneous polynomial of degree $n-6$ in x, y, z, and contains $\frac{1}{2}(n-5)(n-4)$ coefficients If, therefore, such a polynomial ψ satisfies both these equations we must have $(n-5)(n-4)$ homogeneous linear equations between the $\frac{1}{2}(n+1)(n+2)$ coefficients of ψ, i.e , between their $\frac{1}{2}(n+1)(n+2)-1$ mutual ratios. Except in exceptional cases these $(n-5)(n-4)$ equations are linearly independent. Hence, in order that the equations (13a) and (13b) may have a common solution which is a homogeneous polynomial of degree n in x, y, z, we must have

$$(n-5)(n-4)\leqq\tfrac{1}{2}(n+1)(n+2)-1,$$

or

$$n^2-21n+40\leqq 0. \quad (14)$$

Therefore n must lie between 19 and 2. We reject the condition $n>2$, because the equations (13a) and (13b) are at once seen to be satisfied by polynomials of degrees less than or equal to 5.

As the only common analytic solution of the equations (13a) and (13b) is a polynomial of degree not higher than 18 in x, y, z, the equations (12) show that χ can at most be a polynomial of degree 16 in x, y, z. The degree of this polynomial cannot be raised by adding to χ the part corresponding to the state of plane stress, for we have seen (Ghosh, 1942) that χ is then a polynomial of degree not higher than 6.

As these restricted values of χ and ψ give rise to a particular distribution of tractions on the cylindrical edge of the plate, we cannot have in general a state of generalized plane stress in the plate when the stress resultants and stress couples are arbitrarily prescribed on the rim.

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HYPERCUBES OF STRENGTH 'd' LEADING TO CONFOUNDED DESIGNS IN FACTORIAL EXPERIMENTS

By

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Introduction

In a previous paper (Rao, 1945) the author has defined certain combinatorial arrangements called Hypercubes of strength d and has shown that they are derivable from finite geometrical configurations. It is also observed that the existence of a hypercube of strength d leads to confounded designs in the case of symmetrical factorial experiments in which all interactions of the factors up to the order $(d-1)$ are completely preserved. In particular, when $d = 2$ the hypercube becomes a Latin cube of the *first order* defined by Kishen (1942).

In an elegant paper R. A. Fisher (1945), using the properties of groups and hypercubes of strength 2, has given a system of confounded designs in the case of the general factorial experiments preserving all the first order interactions and main effects.

The object of the present paper is to pursue the methods developed in my previous paper and

(a) prove certain optimum properties connected with the arrangements of a hypercube of strength d;

(b) systematise the construction of hypercubes;

(c) discuss simple and quick methods of arriving at confounded designs in the case of symmetrical factorial experiments;

(d) and derive a system of confounded designs in the case of symmetrical and asymmetrical factorial experiments using the arrangements of hypercubes.

2. Hypercubes of strength 'd'

A hypercube of strength d is defined as follows. Let there be m factors $A_1, A_2, \ldots, A_m$ each of which can assume s different values. We define an ordered set $(i_1, i_2, \ldots, i_m)$ as a combination of m factors obtained by the selection of i_1-*th*, i_2-*th* ... values of the first, second, ..., factors respectively. There are s^m such combinations of which a subset of s^t combinations may be called a (m, s, t) array.

An (m, s, t) array is said to be of strength d if all combinations of any d of the m factors occur an equal number ($= s^{t-d}$) *of times*. An array of strength d represented by (m, s, t, d) is, alternatively, called a hypercube of strength d.

In the case $t = 2$, $d = 2$ the arrangement $(m, s, 2, 2)$ corresponds to the existence of $(m-2)$ orthogonal Latin squares of side s and gives an alternative method of representing $(m-2)$ orthogonal squares. In the case of higher values of t and d the definition of a hypercube as a set of ordered combinations avoids the geometrical or any configurational aids in representing them.

Hypercubes of the form (m, s, t, d) may not exist for any assigned set of the

parameters, but an important problem is the construction of hypercubes for the maximum possible value of m for an assigned s, t and d.

3. Hypercubes and finite geometries

Let $PG(t, s)$ represent a finite projective geometry of t dimensions with $(s+1)$ points on a line as defined by Veblen and Bussey (1906). The configuration for a finite Euclidean geometry $EG(t, s)$ of t dimensions is obtained from $PG(t, s)$ by omitting one $(t-1)$ flat and all the points lying on it. This flat may be termed as the flat at infinity.

There are $(s^{t+1}-1)/(s-1)$ points in $PG(t, s)$ of which $(s^t-1)/(s-1)$ lie on the $(t-1)$ flat at infinity. The number of points in $EG(t, s)$ is, therefore, s^t. These are called finite points.

There are $(s^t-1)/(s-1)$, $(t-2)$ flats lying on the $(t-1)$ flat at infinity and through each flat there pass, excluding the flat at infinity, s, $(t-1)$ flats each containing an exclusive set of s^{t-1} finite points. These s, $(t-1)$ flats are called a pencil of parallel flats emanating from a $(t-2)$ flat as their vertex at infinity.

Corresponding to the $(s^t-1)/(s-1)$, $(t-2)$ flats on the $(t-1)$ flat at infinity there are $(s^t-1)/(s-1)$ pencils which may be identified with factors. The s parallel flats in a pencil may be identified with the s levels of a factor. Through each finite point there pass one flat of each pencil and hence a finite point may be identified by the nature flats passing through it or the same as a combination of the factors.

If we write down the s^t combinations corresponding to s^t finite points, they constitute the arrangement $(m, s, t, 2)$ of a hypercube of strength 2 where $m = (s^t-1)/(s-1)$ for any two $(t-1)$ flats belonging to two different pencils intersect in s^{t-2} finite points. Hence any combination of any two factors occurs in s^{t-2} combinations, thus satisfying the requirements of a hypercube of strength 2.

We can consider the s^t finite points as s^t elements in which case s parallel flats each containing s^{t-1} points define $(s-1)$ contrasts built out of s^t elements. Since any two flats belonging to two different pencils intersect in s^{t-2} finite points it follows that contrasts arising out of any two pencils are orthogonal. Since only (s^t-1) contrasts are possible with s^t elements and the $(s^t-1)/(s-1)$ pencils define (s^t-1) independent contrasts, it follows that no more pencils giving rise to orthogonal contrasts are possible. Thus the maximum value of m for a given s, t, and $d = 2$ is $(s^t-1)/(s-1)$.

If we want to get a hypercube of strength 3, it is necessary that three flats belonging to three different pencils chosen from the set of $(s^t-1)/(s-1)$ pencils intersect in s^{t-3} finite points. Any two vertices corresponding to two pencils intersect in a $(t-3)$ flat at infinity. If a third vertex is chosen passing through this $(t-3)$ flat then the flats of this pencil will have the same set of s^{t-2} finite points in common with the intersection of two flats belonging to pencils already chosen. On the other hand if the third vertex is chosen such that it does not pass through the intersection of the other two then three flats belonging to the three different pencils chosen intersect in s^{t-3} finite points. The fourth vertex has to be chosen such that it does not pass through the

intersections of any two vertices already chosen and so on. It is easy to see that the conditions for a hypercube of strength 3 are satisfied if the vertices so chosen are identified with factors, The optimum value of m for a hyperenbe of strength 3 is the number of vertices that can be chosen from the flat at infinity satisfying the above properties.

If we want to construct a hypercube of strength d we have to choose pencils such that none of the vertices belonging to these pencils pass through the intersection of two or more of $(d-1)$ vertices already chosen. In this case any set of d flats belonging to d different pencils intersect in s^{t-d} finite points thus satisfying the properties of a hypercube of strength d. The optimum value of m for an (m, s, t, d) array is the number of such vertices that can be chosen from the flat at infinity. There does not seem to be an easy method of arriving at a general expression for this optimum value but some inequalities connecting m, s, t, d may be available.

As we are using finite geometrical configurations of dimensions higher than 2, it follows that construction of hypercubes as developed above is available when and only when s is a prime or a prime power. Even in the case of $t = 2$, the known finite geometrical configurations correspond to prime or prime power values of s. Also the problem of getting all possible hypercubes remains unsolved.

4. Construction of hypercubes with the analytical representation of $PG(t, s)$

If s is a prime or a prime power there exists a Golois field $GF(s)$ with s elements which may be represented by $\alpha_1 = 0, \alpha_2, \alpha_3, \ldots, \alpha_s$. Any finite point of $PG(t, s)$ may be represented by an ordered set

$$(x_0, x_1, \ldots, x_t) \tag{4.1}$$

of $(t+1)$ coordinates where the first coordinate x_0 is the unit element in $GF(s)$ and $x_1, \ldots, x_t$ are any elements of $GF(s)$. In particular, the coordinate x_0 may be dropped when we are referring to finite points. There are s^t possible combinations of the type (4.1) which correspond to the s^t finite points. Any point on the $(t-1)$ flat at infinity may be represented by

$$(x_0, x_1, \ldots, x_t) \tag{4.2}$$

where $x_0 = 0$ and the first nonnull element in the ordered series is a unit element. There are $(s^t-1)/(s-1)$ such combinations possible leading to the correspondence with $(s^t-1)/(s-1)$ points on the flat at infinity.

The equation to the $(t-1)$ flat at infinity is given by $x_0 = 0$ and any $(t-1)$ flat has an equation of the form

$$a_0x_0 + \cdots + a_tx_t = 0 \tag{4.3}$$

and any $(t-2)$ flat at infinity may be represented by

$$x_0 = 0, \quad a_1x_1 + \cdots + a_tx_t = 0. \tag{4.4}$$

The s, $(t-1)$ flats of the pencil corresponding to this $(t-2)$ flat are given by

$$-\alpha_i x_0 + a_1x_1 + \cdots + a_tx_t = 0, \quad (i = 1, 2, \ldots, s). \tag{4.5}$$

The flat with $-\alpha_i$ as the coefficient of x_0 may be identified by α_i which may correspond to the i-th level of a factor. The pencil of flats (4.5) or the factor corresponding to it may be identified by the set $[a_1, a_2, \ldots, a_t]$ defining the vertex of the pencil (4.5) at infinity. The factors represented by $[a_1, a_2, \ldots, a_t]$ and $[\sigma a_1, \sigma a_2, \ldots, \sigma a_t]$ where $\sigma \neq 0$ are, according to our identification, the same.

We have, now, to consider each finite point and list down the nature of flats passing through it after numbering the vertices serially. The $(t-2)$ flats at infinity may all be obtained as $x_0 = 0$ and a combination of flats

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_t = 0 \tag{4.6}$$

which may be identified by factors $A_1, A_2, \ldots A_t$. The flats corresponding to the vertex $x_0 = 0$, $x_i = 0$ or the factor A_i are given by considering only finite points,

$$-\alpha_j + x_i = 0, \quad j = 1, 2, \ldots, s. \tag{4.7}$$

If the finite point $(\beta_1, \beta_2, \ldots, \beta_t)$ gives the levels $\xi_1, \xi_2, \ldots, \xi_t$ for the factors $A_1, A_2, \ldots, A_t$ then the level for the factor represented by $[a_1, a_2, \ldots, a_t]$ with the convention that the first nonnull value of a is unity, is given by $a_1\xi_1 + a_2\xi_2 \ldots + a_t\xi_t$. Also if the levels of a factor for the finite points $(1, 0, 0, \ldots 0)$, $(0, 1, 0, \ldots, 0)$, $(0, 0, 1, \ldots, 0), \ldots (0, 0, 0, \ldots, 1)$ are given by $\eta_1, \eta_2, \ldots \eta_t$ then the levels of that factor for the finite point $(\beta_1, \beta_2, \ldots, \beta_t)$ is given by $\beta_1\eta_1 + \cdots + \beta_t\eta_t$. Thus, it appears that the combinations arising out of a chosen set of vertices (of which the sub set (4.6) may always be chosen) can be simply generated from the scheme given below.

Method of constructing Hypercubes

finite point	levels of chosen factors		
	$A_1A_2 \ldots A_t$	$[a_1a_2 \ldots a_t]$	$\ldots$
$(0\ 0 \ldots 0)$	$0\ 0 \ldots 0$	0	
$(1\ 0 \ldots 0)$	$1\ 0 \ldots 0$	a_1	
$(0\ 1 \ldots 0)$	$0\ 1 \ldots 0$	a_2	$\ldots$
$\ldots\ \ldots$	$\ldots\ \ldots$	$\ldots$	
$(0\ 0 \ldots 1)$	$0\ 0 \ldots 1$	a_t	
$(\beta_1\beta_2 \ldots \beta_t)$	$\beta_1\beta_2 \ldots \beta_t$	$\beta_1a_1 + \cdots + \beta_ta_t$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$

The column of finite points may be omitted as the selections of the factors $A_1, A_2, \ldots, A_t$ are same as the combinations for finite points.

Any position can thus be filled up by the scalar product as of vectors of the representations of finite points and the factors as ordered set of elements.

Illustrative example. If $t = 3$, $s = 3$, there are 13 lines on the plane at infinity. Taking the residue classes mod 3 for $GF(3)$ we can write down the 13 lines and identify them by latin letters.

$$\begin{array}{llll} A,\ x_1 = 0, & D,\ x_1 + x_2 + x_3 = 0, & H,\ x_1 + x_2 = 0, & K,\ x_1 + 2x_2 = 0 \\ B,\ x_2 = 0, & E,\ x_1 + 2x_2 + 2x_3 = 0, & I,\ x_2 + x_3 = 0, & L,\ x_2 + 2x_3 = 0 \\ C,\ x_3 = 0, & F,\ x_1 + 2x_2 + x_3 = 0, & J,\ x_1 + x_3 = 0, & M,\ x_1 + 2x_3 = 0 \\ & G,\ x_1 + x_2 + 2x_3 = 0. & & \end{array}$$

If we take these 13 vertices and consider the 13 pencils passing through them we get an arrangement for (13, 3, 3, 2). But to construct a hypercube of strength 3 we have to choose the vertices as indicated above. One such selection is given by

$$A,\ x_1 = 0, \quad C,\ x_3 = 0, \quad B,\ x_2 = 0,$$

$$D,\ x_1 + x_2 + x_3 = 0$$

showing that the optimum value of $m = 4$. The 27 combinations are listed below

The Hypercube (4, 3, 3, 3)

A	B	C	D	A	B	C	D	A	B	C	D
0	0	0	0	1	1	0	2	2	0	1	0
1	0	0	1	0	2	2	1	1	1	1	0
0	1	0	1	2	0	2	1	2	2	2	0
0	0	1	1	2	2	0	1	1	2	2	2
2	0	0	2	0	1	2	0	2	2	1	2
0	2	0	2	1	2	0	0	2	1	2	2
0	0	2	2	1	0	2	0	2	1	1	1
0	1	1	2	0	2	1	0	1	1	2	1
1	0	1	2	2	1	0	0	1	2	1	1

In the case $s = 2$, $t = 3$, the maximum value of $m = 7$. The lines at infinity are

$$\begin{array}{lll} A,\ x_1 = 0 & D,\ x_1 + x_2 = 0 & \\ B,\ x_2 = 0 & E,\ x_2 + x_3 = 0 & G,\ x_1 + x_2 + x_3 = 0. \\ C,\ x_3 = 0 & F,\ x_1 + x_3 = 0 & \end{array}$$

The Hypercube (7, 2, 3, 2)

A	B	C	D	E	F	G
0	0	0	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	1	0	1	1	1
0	1	1	1	0	1	0
1	1	0	0	1	1	0
1	0	1	1	1	0	0
1	1	1	0	0	0	1

If we want to construct a hypercube of strength 3, with $t = 3$ and $s = 2$ we may choose the following lines.

$$A,\ x_1 = 0, \quad B,\ x_2 = 0, \quad C,\ x_3 = 0, \quad D,\ x_1+x_2+x_3 = 0$$

The Hypercube (4, 2, 3, 3)

A	B	C	D
0	0	0	0
0	0	1	1
0	1	0	1
1	0	0	1
0	1	1	0
1	0	1	0
1	1	0	0
1	1	1	1

The properties of these cubes remain the same even if one or more factors are omitted.

5. A system of confounding for s^m experiment.

If there are m factors each at s levels, they give rise to s^m treatment combinations. Identifying the s levels of a factor with the s elements of $GF(s)$ we may define the main effects and interactions as given by Bose and Kishen (1942).

If $(x_1, x_2, \ldots, x_m)$ represent a combination of the m factors there are s^{m-1} combinations satisfying a linear equation of the type

$$b_1x_1+\cdots+b_mx_m-\alpha_i = 0 \tag{5.1}$$

where $b_1, b_2, \ldots b_m$ and α_i are elements of $GF(s)$. By keeping b's fixed and allowing α's to assume all values in $GF(s)$ we get s mutually exclusive sets of s^{m-1} combinations

in each, thus covering the totality of the sets. We define the $(s-1)$ contrasts between these sets as the contrast set $[b_1, b_2, \ldots, b_m]$. The rectangular bracket is used to distinguish the contrast set from a treatment combination. It is easy to see that the contrast sets represented by $[b_1, b_2, \ldots, b_m]$ and $[\sigma b_1, \sigma b_2, \ldots, \sigma b_m]$ are the same. This combination may, as well, be said to carry $(s-1)$ degrees of freedom.

Accepting this notation we may list the contrast sets belonging to main effects and interactions of the factors denoted by A, B, C,... as follows

Contrast set		Degree of freedom of each contrast set	Effect to which the set belongs
$[b_1, 0, \ldots, 0]$	$(b_1 \neq 0)$	$(s-1)$	A
$[b_1, b_2, \ldots, 0]$	$(b_1 \neq 0,\ b_2 = \alpha_2, \ldots, \alpha_s)$	$(s-1)$	AB
$[b_1, b_2, b_3, \ldots, 0]$	$(b_1 \neq 0,\ b_2,\ b_3 = \alpha_2, \ldots, \alpha_s)$	$(s-1)$	ABC
...			

In general if $b_i, b_j, \ldots, b_u$, are not zero then the contrast set belongs to the interaction of i-th, j-th, ..., u-th factors.

Corresponding to an interaction of k factors there are $(s-1)^{k-1}$ contrast sets carrying $(s-1)^k$ degrees of freedom. Since there are ${}^mC_k$ selections possible for interactions of k factors the total degrees of freedom carried by all the main effects and interactions are

$$\sum {}^mC_k(s-1)^k = s^m - 1, \tag{5.2}$$

thus covering the $s^m - 1$ contrasts among the s^m combinations.

The problem of confounding is the splitting of these s^m combinations in s^{m-t} sets of s^t combinations each such that the $(s^{m-t}-1)$ contrasts arising out of these sets may be identified in sets of $(s-1)$ contrasts with those defined above. Or in other words the $(s^{m-t}-1)$ degrees of freedom should break up to $(s^{m-t}-1)/(s-1)$ sets of $(s-1)$ degrees of freedom each such that these sets belong to the contrast sets enumerated above. Such arrangements lead to confounded designs for the s^m experiment in blocks of s^t plots obtained by assigning each of the above s^{m-t} sets to a block.

If $[b_1, b_2, \ldots, b_m]$ and $[c_1, c_2, \ldots, c_m]$ are two contrast sets that are confounded then the combinations in the set containing $(0, 0, \ldots, 0)$ must satisfy the equations

$$\left.\begin{aligned} b_1x_1 + \cdots + b_mx_m &= 0 \\ c_1x_1 + \cdots + c_mx_m &= 0 \end{aligned}\right\} \tag{5.3}$$

Hence it follows that these combinations satisfy the equation

$$(b_1 + \alpha c_1)x_1 + \cdots + (b_m + \alpha c_m)x_m = 0 \tag{5.4}$$

where α is an element of $GF(s)$. Hence the contrast sets confounded form a *Subspace of Contrast Sets.* All the contrast sets generated from a group of contrast sets by linear combinations (as of vectors) are said to form a space of contrast sets. If this space covers only a part of the totality of contrast sets then it s termed as a subspace.

If we want to confound a given subspace of $(s^{m-t}-1)/(s-1)$ contrast sets then we find out the $(m-t)=\lambda$ independent contrast sets from which the totality of $(s^{m-t}-1)/(s-1)$ contrast sets can be generated. If the independent contrast sets are

$$[b_{1i}, b_{2i}, \ldots, b_{mi}], \quad i=1, 2, \ldots, \lambda. \tag{5.5}$$

then a block of the confounded design is obtained from the combinations satisfying the set of equations

$$\begin{array}{c} b_{11}x_1+\cdots+b_{m1}x_m-\alpha_{i_1}=0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ b_{1\lambda}x_1+\cdots+b_{m\lambda}x_m-\alpha_{i_\lambda}=0. \end{array} \tag{5.6}$$

By allowing $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_\lambda}$ to assume all values in $GF(s)$ we get s^λ sets of equations each giving rise to a block. The totality of these blocks leads to a design confounding the given subspace of contrast sets.

A quick method of deriving these blocks is as follows. We find the combinations obtained from the equations

$$b_{1i}x_1+\cdots+b_{mi}x_m=0, \quad i=1, 2, \ldots, \lambda \tag{5.7}$$

The block containing these combinations is called a *keyblock*. If $(d_1, d_2, \ldots, d_m)$ is any combination in any other block then all combinations of that block are obtained from

$$(d_1+\xi_1, \ d_2+\xi_2, \ \ldots, \ d_m+\xi_m) \tag{5.8}$$

where $(\xi_1, \xi_2, \ldots, \xi_m)$ is a combination in the key block. The method of deriving the totality of blocks is to start with any combination not used and build up the rest by additions of combinations obtained in the key block in the manner shown in (5.8) for these constitute solutions to a set of equations (5.8).

The combinations in the key block form a subspace of treatment combinations, subspace being defined in a manner similar to that in the case of contrast sets, since they constitute the set of solutions to a system of linear equations (given in 5.7). This enables us to find the confounded contrast subspace when the key block is given. Taking the combinations $(\xi_{1i}, \ldots, \xi_{mi})$ from the key block we form the set of equations

$$\xi_{1i}b_1+\cdots+\xi_{mi}b_m=0, \quad i=1, 2, \ldots, k \tag{5.9}$$

where k is the size of the key block. This gives us a set of solutions giving the subspace of confounded contrast sets. The contrast subspace confounded as found from (5.9) may be said to be orthogonal to the treatment subspace of the key block.

6. Hypercubes as key blocks

If we want an arrangement in blocks of size s^t for the maximum possible number of factors preserving all interactions up to the order $(d-1)$ then the contrast subspace orthogonal to the treatment subspace of the key block must contain no contrast set involving d or less than d non-null elements.

We shall now prove that a hypercube (m, s, t, d) of strength d as constructed in Section 4 is a subspace of treatment combinations generated from t independent

combinations and the orthogonal contrast space contains no contrast set involving less than $(d+1)$ nonnull elements. As the hypercube contains the combination $(0, 0, \ldots, 0)$ it follows that it can be used as a key block for getting a confounded design in which none of the interactions up to d factors are confounded.

The first part of the result comes out from the nature of the construction of hypercubes of strength d as given in Section 4. To prove the second part of the result we need only show that there does not exist a set of quantities $b_{i_1}, b_{i_2}, \ldots, b_{id}$ such that

$$b_{i_1}x_{i_1}+\cdots+b_{id}x_{id} = 0 \tag{6.1}$$

for all sets $(x_{i_1}, \ldots, x_{id})$ of the combinations of the $i1$-th, ..., id-th factors chosen from the key block. Since each such combination occurs s^{t-d} times it follows that the number of different equations of the type (6.1) is s^d corresponding to the s^d combinations of d factors. Since, in particular, the combinations of d factors $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$ ..., etc. are possible it follows that each of the b's must be zero. Hence no contrast can involve less than $(d+1)$ nonnull elements. Given the hypercube of strength d constructed out of the t generators

$$(g_{1i}, g_{2i}, \ldots, g_{mi}), \quad i = 1, 2, \ldots, t \tag{6.2}$$

the equations giving the contrast space are

$$b_1 g_{1i}+\cdots+b_m g_{mi} = 0, \quad i = 1, 2, \ldots, t. \tag{6.3}$$

There are $(m-t)$ independent solutions giving rise to $(s^{m-t}-1)/(s-1)$ contrast sets which are totally confounded. Thus the contrasts confounded by using a hypercube as a key block can be ascertained.

The hypercubes of strength d as constructed in Section 4 lead to confounded designs for the maximum possible number of factors in blocks of size s^t preserving all main effects and interactions containing d and lesser number of factors. By omitting one or more columns from the hypercube we get designs satisfying the same property with lesser number of factors.

Illustrative example. As an example we can find the contrasts confounded in using the hypercube (4, 3, 3, 3) as keyblock for a 3^4 design in blocks of 3^3 confounding only interactions containing four factors.

The generators of (4, 3, 3, 3) are

A	B	C	D
1	0	0	1
0	1	0	1
0	0	1	1

The only orthogonal contrast is [2, 2, 2, 1].

Similarily we can find the contrasts confounded in using the hypercube (7, 2, 3, 2) as key block for a 2^7 design in blocks of 2^3 plots. The generators are

A	B	C	D	E	F	G
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	1	0	1	1	1.

The orthogonal space can be immediately written down as

1	1	0	1	0	0	0
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	1

which gives rise to the interactions

$$ABD, \quad BCE, \quad ACF, \quad ABCG$$

and those derived from them.

7. Hypercubes and asymmetrical factorial designs

An asymmetrical factorial design (Nair and Rao, 1941) may be defined in the case of two factors as follows. If the levels of these factors denoted by A_1 and B_1 are u and s respectively, then there are su treatment combinations identified by the ordered sets

$$(a_i b_j); \quad i = 1, 2, \ldots, u; \quad j = 1, 2, \ldots, s. \tag{7.1}$$

A set of $k < su$ combinations is called a block. If there are b sets of k combinations each such that

(a) each combination is used r times,

(b) the combinations $(a_i b_j)$ and $(a_h b_m)$ occur together in λ_{11} blocks if $i \neq h$, $j \neq m$; λ_{10} blocks if $i \neq h$, $j = m$; λ_{01} blocks if $i = h$, $j \neq m$ and zero times if $i = h$ and $j = m$,

then the arrangement is said to be an asymmetrical factorial design.

Some of these arrangements are derivable from hypercubes of strength 2 as derived in Section 4. We take u of the factors in the hypercube and indentify them with the u levels of the factor A_1 and the s levels of each of the factors with s levels of the second factor B_1. Then each finite point gives rise to u combinations. These combinations are taken as a block. There are as many blocks as there are finite points

Since the hypercube $(m, s, t, 2)$ is of strength 2 it follows that corresponding to any two different levels of A_1 all possible combinations of the levels of the second factor B_1 occur an equal number s^{t-2} times which gives the result that $\lambda_{11} = \lambda_{10} = s^{t-2}$. Since no level of A_1 is repeated in a block it follows that $\lambda_{01} = 0$ Since the property of the hypercube of strength 2 is retained even if some of the factors are omitted it follows that the levels of the first factor is less than or equal to m. Since a hypercube

$(m', s, t-1, 2)$ can also be constructed it follows that the minimum values of λ parameters for the design $u \times s$ are as given above if

$$m' = (s^{t-1}-1)/(s-1) < u \leq (s^t-1)/(s-1) = m. \tag{7.2}$$

If $u \leq (s^{t-1}-1)/(s-1)$ then the parameters $\lambda_{11} = \lambda_{10} = s^{t-3}$ may be used and the design is derivable from $(m', s, t-1, 2)$.

An illustrative example. With the help of the hypercube (7, 2, 3, 2) given in Section 4 we can get two dimensional designs $u \times 2$ where $3 < u \leq 7$ with the system of parameters $\lambda_{11} = \lambda_{10} = 2$, $\lambda_{01} = 0$. The factors A, B, C, D, E, F, G may be identified with the levels 1, 2, . . . , 7 of A_1.

Design for 7×2 giving designs for $u \times 2$.

levels of the first factor	A 1	B 2	C 3	D 5	E 6	F 7	G 4	block no.
levels of the second factor	0	0	0	0	0	0	0	1
	1	0	0	1	0	1	1	2
	0	1	0	1	1	0	1	3
	0	0	1	0	1	1	1	4
	0	1	1	1	0	1	0	5
	1	1	0	0	1	1	0	6
	1	0	1	1	1	0	0	7
	1	1	1	0	0	0	1	8

By omitting levels corresponding to G, F, etc., we get designs for $u \times 2$, where $3 < u \leq 7$ with $\lambda_{11} = 2 = \lambda_{10}$, $\lambda_{01} = 0$.

If we omit the levels 5, 6, 7 and the blocks 1 and 8 we get the design for 4×2 in blocks of 4 plots with $\lambda_{11} = 2$, $\lambda_{10} = 1$, $\lambda_{01} = 0$.

This leads us to search for designs with $\lambda_{11} = s^{t-1}$, $\lambda_{10} = s^{t-2}-1$ and $\lambda_{01} = 0$ in the general case.

We take a hypercube $(m, s, t, 2)$ of strength 2 and consider the finite point $(1, 0, \ldots, 0)$. These are s^{t-1} factors which have the selection 1. These very factors will have the selection α_i for the point $(\alpha_i, 0, \ldots, 0)$. We have to retain only these factors and omit the blocks corresponding to the s finite points $(\alpha_i, 0, \ldots, 0)$; $i = 1, 2, \ldots, s$. This, evidently, gives us the design for the set of parameters $u = s^{t-1}$, $\lambda_{11} = s^{t-2}$, $\lambda_{10} = s^{t-2}-1$ and $\lambda_{01} = 0$. Even if some factors are omitted from this the λ parameters remain the same. Hence we get the following results for the two dimensional $u \times s$ design in blocks of u plots.

limits of u	values of λ_{11}	λ_{10}	λ_{01}
$(s^{t-1}-1)/(s-1) < u \leq (s^{t}-1)/(s-1)$	s^{t-2}	s^{t-2}	0
$(s^{t-1}-1)/(s-1) < u \leq s^{t-1}$	s^{t-2}	$s^{t-2}-1$	0

The extension of these results to higher dimensional cases will be given elsewhere.

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ON THE POWER FUNCTION OF STUDENTISED D^2-STATISTIC

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1. *Definition and notation.* Let $x_1, x_2, \ldots, x_p$ be p random variables obeying the elementary probability law:

$$f(x_1, x_2, \ldots, x_p, \theta_1, \theta_2, \ldots, \theta_k)dx_1\,dx_2 \ldots dx_p. \tag{1}$$

The vector $(x_1, x_2, \ldots, x_p)$ will be represented in a p-dimensional Euclidean space W by a point E while $(\theta_1, \theta_2, \ldots, \theta_k)$ will be represented by a point θ in the k-dimensional space Ω, called the parametric space. The test of a hypothesis $H_0(\theta=\theta^0)$ consists in choosing a region w in W called the critical region such that if we meet with a point in w, we reject H_0. The chances of rejecting $H_0(\theta = \theta^0)$ and $H(\theta = \theta^1)$ by adopting this rule of procedure are respectively the size of the critical region and the power of the test with respect to the alternative H, given by

$$P(E \in w/H_0 \text{ or } H) = \int_w f(x/\theta^0 \text{ or } \theta')dx. \tag{2}$$

The p-variate normal population Π_1 is given by

$$f(x/\theta)dx = \text{const.} \times \exp\{-\tfrac{1}{2} \sum_{i,j=1}^{p} \alpha^{ij}(x_i - m_i)(x_j - m_j)\}dx_1\,dx_2 \ldots dx_p, \tag{3}$$

where m_j is the expectation of x_j and α^{ij} is the (i, j)th element in the inverse of the dispersion matrix $\|\alpha_{ij}\|$. If we have a sample of size n_1 from (3) and size n_2 from another p-variate normal population Π_2 denoted by x''s and m''s but with the same dispersion matrix, then to test

$$H_0(m_1 = m_1',\ m_2 = m_2',\ \ldots,\ m_p = m_p') \tag{4}$$

whatever may be $\|\alpha_{ij}\|$, the statistic D^2 has been proposed:

$$D^2 = (1/p) \sum_{i,j=1}^{p} c^{ij}(\bar{x}_i - \bar{x}_i')(\bar{x}_j - \bar{x}_j') \tag{5}$$

where $\bar{x}_i$ is the sample mean of the i-th character in Π_1 and c^{ij} is the (i, j)th element in the inverse of the sample dispersion matrix $\|c_{ij}\|$,

$$n\,c_{ij} = (n_1+n_2-2)c_{ij} = \sum_{k=1}^{n_1} (x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j) + \sum_{k=1}^{n_2} (x_{ik}' - \bar{x}_i')(x_{jk}' - \bar{x}_j'). \tag{6}$$

Let us define Δ^2 by

$$\Delta^2 = (1/p) \sum_{i,j=1}^{p} \alpha^{ij}(m_i - m_i')(m_j - m_j'). \tag{7}$$

However, instead of D^2 we will consider the equivalent statistic D_1^2 given by

$$D_1^2 = \frac{c^2D^2}{n+c^2D^2} \quad \text{where} \quad c^2 = p.\frac{n_1n_2}{n_1+n_2} \tag{8}$$

and define

$$\beta^2 = \frac{c^2\Delta^2}{2}. \tag{9}$$

D_1^2 will vary between 0 and 1 and we will study the optimum properties of the critical region w_1 such that inside w_1, $D_1^2 \geq D_0^2$(constant). The size and the power of this critical region w_1 are given respectively by (Roy, 1938)

$$P(E \in w_1/\beta = 0) = \frac{\Gamma\{(n+1)/2\}}{\Gamma(p/2)\Gamma\{(n-p+1)/2\}} \int_{D_0^2}^{1} (1-D_1^2)^{(n-p-1)/2} (D_1^2)^{(p-2)/2} dD_1^2, \tag{10}$$

$$P(E \in w_1/\beta \neq 0) = \frac{\Gamma\{(n+1)/2\}}{\Gamma(p/2)\Gamma\{(n-p+1)/2\}} \int_{D_0^2}^{1} (1-D_1^2)^{(n-p-1)/2} (D_2^2)^{(p-2)/2}$$
$$\times {}_1F_1\left(\frac{n+1}{2}, \frac{p}{2};\ D_1^2\beta^2\right) dD_1^2. \tag{11}$$

2. *Validity of a test.* When the hypothesis to be tested is a composite one, the critical region should be similar to the sample space, *i.e.*, the region should allow probability statements, independent of the parameters about which no knowledge is presumed. Moreover, since by a test all admissible departures from the hypothesis H are to be detected, it is necessary that the power function of the critical region to be considered should depend on admissible alternatives. Thus we may define the validity of a test:

Definition. A critical region w_0 similar to the sample space makes a valid test of the hypothesis H_0, if $P(E \in w_0/H) = \phi(H)$, where H stands for any admissible alternative hypothesis.

To test the hypothesis in (4), the similar region w^i corresponding to the test for the equality of means of the i-th character in the two populations, *i e.*, Fisher's 't' for the i-th character might be taken. But if we consider the power function of w^i it will be found

$$\begin{aligned} &P\{E \in w^i/H(m_1 = m_1', \ldots, m_i \neq m_i', m_{i+1} = m_{i+1}', \ldots)\} \\ &= P\{E \in w^i/H(m_1 \neq m_1', \ldots, m_i \neq m_i', \ldots, m_p \neq m_p')\} \\ &= \phi(m_i - m_i'). \end{aligned}$$

and as such w^i does not supply a valid test. Since Δ^2 in (7) is a positive definite quadratic form and vanishes when and only when $m_i = m_i'$ for all i's, the region w_1 will be seen to make a valid test of the hypothesis in (4).

It may be pointed out that the tests obtained by 'Likelihood Criteria' are all valid whenever they give similar regions (Wald, 1943).

3. *The Resultant Power and Success rate of a test.* While testing H_0, we will, in fact, come across H_0 and all alternative H's. The choice of a critical region should be based on the consideration that we want to make successful statements regarding the correctness and incorrectness of H_0 in as large a percentage of cases as possible. To put this symbolically, let us assume an integral probability law for the *a priori* distribution of H's given by

$$P\{\theta_i \leq \theta_i^0,\ (i = 1, 2, \ldots, p)\} = P(\theta^0) \tag{12}$$

which has a 'saltus' of magnitude γ_0 at the point θ^0 corresponding to H_0. The set of points in Ω other than θ^0 will be denoted by Ω'. Let w be a valid critical region of size α. Then the resultant power of the test is defined by (Neyman and Pearson, 1933),

$$1-\gamma_0-P_{11}(w) = \int_{\Omega'} P(E \in w/\theta)dP(\theta) \tag{13}$$

and the chance of success will be given by (Mises, 1942),

$$S = (1-\alpha)\gamma_0 + \int_{\Omega'} P(E \in w/\theta)dP(\theta). \tag{14}$$

The chance of sure success S_0 (Mises, 1942) is the lower bound of S with respect to $P(\theta)$. If the lower bound of $P(E \in w/\theta) = \delta$ when θ changes, then it is easily seen that S_0 is the smaller of δ and $(1-\alpha)$. Moreover, when $P(E \in w/\theta)$ is a continuous function of θ in an interval of θ containing θ^0, $\alpha \geq \delta$ and the largest possible value of S is $\frac{1}{2}$ when $\alpha = \delta = \frac{1}{2}$.

A critical region w_0 of given size will be the best with the *a priori* law (12), if no other critical region of the same size has its resultant power greater than that of w_0. If the size of the critical region is not fixed, it will be said to have optimum size provided α is so chosen as to make the chance of success in (14) maximum; and among critical regions of optimum size the one with the highest chance of success for the law (12) will be regarded as the best.

The necessary and sufficient condition for the existence of the best critical region independently of any *a priori* probability law is the existence of a uniformly most powerful test.

Since the *a priori* law is seldom known and the existence of a uniformly most powerful test is a rarity, the choice falls on a test having the highest chance of sure success which is the same as a uniformly unbiassed critical region of size $\frac{1}{2}$.

Now it is easy to see the test w_1 of Section 1 is uniformly unbiassed for every value of α. Let us take $z = D_1^2$ and $\lambda = \beta^2$ in (11) which becomes

$$P(E \in w_1/\lambda \neq 0) = \frac{\Gamma\{(n+1)/2\}\, e^{-\lambda}}{\Gamma(p/2)\Gamma\{(n-p+1)/2\}} \int_{z_0}^{1} z^{(p-2)/2}(1-z)^{(n-p-1)/2} \cdot {}_1F_1\left(\frac{n+1}{2}, \frac{p}{2}, \lambda z\right)dz$$

$$= \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \eta_i, \tag{15}$$

where

$$\eta_i = \frac{1}{B\{(p+2i)/2, (n-p-1)/2\}} \int_{z_0}^{1} z^{(p+2i-2)/2}(1-z)^{(n-p-1)/2}\, dz \tag{16}$$

and these η_i's (≤ 1) form a monotonic increasing sequence. Differentiating (15), term by term with respect to λ,

$$\frac{\partial}{\partial \lambda} P(E \in w_1/\lambda) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot \eta_{i+1} - \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \eta_i > 0. \tag{17}$$

Thus $P(E \in w_1/\lambda)$ is a monotonic increasing function of λ and as such the critical region w_1 is uniformly unbiassed.

4. *The Resultant Power of* w_1: To test the hypothesis in (4), no uniformly most powerful region exists but it will be shown that for a fixed dispersion matrix $\|\alpha_{ij}\|$ as well as fixed

$$M_i = \frac{n_1 m_i + n_2 m_i'}{n_1 + n_2}$$

but $\theta_i = m_i - m_i'$ following the *a priori* probability law,

$$\psi(\Delta^2)d\theta_1 d\theta_2 \ldots d\theta_p \tag{18}$$

the region w_1 has greater resultant power than any other region of the same size if the admissible alternatives include all values of θ's such that $\Delta^2 \leq \Delta_0^2$.

For a sample of n_1 individuals from Π_1 and n_2 individuals from Π_2, the density factor can be written as

$$f(x, \theta) = G.\exp[-\tfrac{1}{2}\sum_{i,j=1}^{p} \alpha^{ij}\{\sum_{l=1}^{n_1}(x_{il}-m_i)(x_{jl}-m_j) + \sum_{k=1}^{n_2}(x_{ik}'-m_i')(x_{jk}'-m_j')\}].$$

Write

$$\left.\begin{aligned} M_i &= \frac{n_1 m_i + n_2 m_i'}{n_1+n_2} \qquad \text{and} \qquad X_i = \frac{n_1\bar{x}_i + n_2\bar{x}_i'}{n_1+n_2} \\ a_{ij} &= \sum_l (x_{il}-X_i)(x_{jl}-X_j) + \sum_k (x_{ik}'-X_i)(x_{jk}'-X_j) \end{aligned}\right\} \tag{19}$$

Then

$$f(x, \theta) = G.\exp\Big[-\tfrac{1}{2}\sum_{i,j}\alpha^{ij}\Big\{\sum_l (x_{il}-M_i)(x_{jl}-M_j) + \sum_k (x_{ik}'-M_i)(x_{jk}'-M_j)$$

$$-\frac{n_1 n_2}{n_1+n_2}(\bar{x}_i-\bar{x}_i')(m_j-m_j') - \frac{n_1 n_2}{n_1+n_2}(\bar{x}_j-\bar{x}_j')(m_j-m_j') + \frac{n_1 n_2}{n_1+n_2}(m_i-m_i')(m_j-m_j')\Big\}\Big]$$

$$= G.\exp\Big[-\tfrac{1}{2}\sum_{i,j}\alpha^{ij}\{a_{ij} + (n_1+n_2)(X_i-M_i)(X_j-M_j)\} + \frac{n_1 n_2}{n_1+n_2}\sum_{i,j}\alpha^{ij}(\bar{x}_i-\bar{x}_i')(m_j-m_j') - \tfrac{1}{2}c^2\Delta^2\Big] \tag{20}$$

To test the hypothesis in (4), the similar region must be built out of the intersection of the surfaces

$$\left.\begin{aligned} a_{ij} &= \text{a constant } (i, j = 1, 2, \ldots, p), \\ X_i &= \text{a constant } (i = 1, 2, \ldots, p). \end{aligned}\right\} \tag{21}$$

The critical region w' of size α having greater resultant power than any other region of the same size will be given by

$$\int_{\Delta^2 \leq \Delta_0^2} f(x, \theta)\psi(\Delta^2)d\theta_1 \ldots d\theta_p \geq k\int_{\Delta^2 \leq \Delta_0^2} f(x, \theta^0)\psi(\Delta^2)d\theta_1 \ldots d\theta_p \tag{22}$$

on the surfaces (21).

If we now make a linear transformation of p variates which will, of course, keep invariant (20) in form such that $\alpha_{ij} = 0$ $(i \neq j)$ or $1 (i = j)$, then on the surfaces (21) the left-hand side of (22) will be an increasing function of T^2 where

$$T^2 = \sum_{i=1}^{p}(\bar{y}_i - \bar{y}_i')^2 \tag{23}$$

and y's are the new variates corresponding to x's.

Let us try to evaluate T^2. Consider the sample space W consisting of two orthogonal spaces W_{n_1} and W_{n_2} of n_1 and n_2 dimensions respectively. The sample Σ from population Π_1 will be represented by p points $P_i (i = 1, 2 \ldots, p)$ in W_{n_1}, P_i having the co-ordinates $(x_{i1}, x_{i2}, \ldots, x_{in_1})$. Similarly the sample Σ' from population Π_2 will be

represented by p points $P_i'(i=1, 2, \ldots, p)$ in W_{n_2}. Let OE and OE' be the equiangular lines in W_{n_2} and W_{n_1} respectively. If the feet of the perpendiculars from P_i and P_i' on OE and OE' respectively be denoted by Q_i and Q_i', then let us concentrate on the $2p$ vectors $\overline{OZ}_i$, $\overline{OZ}_i'(i=1, 2, \ldots, p)$ respectively parallel and equal to $\overline{P_iQ}_i$ and $\overline{P_i'Q}_i'$, $2p$ vectors $\overline{OQ}_i$, $\overline{OQ}_i'(i=1, 2, \ldots, p)$ and the equiangular lines OE and OE'.

Find the resultant of $\overline{OQ}_i/\sqrt{n_1}$ and $\overline{OQ}_i'/\sqrt{n_2}$ and denote it by $\overline{OR}_i(i=1, 2, \ldots, p)$. Then the projection $\overline{OY}_i$ of $\overline{OR}_i$ on the external bisector OA of OE and OE' will have the co-ordinates of its end-point as

$$\left(\frac{\bar{x}_i-\bar{x}_i'}{2\sqrt{n_1}}, \ldots, \frac{\bar{x}_i-\bar{x}_i'}{2\sqrt{n_1}}, -\frac{\bar{x}_i-\bar{x}_i'}{2\sqrt{n_2}}, \ldots, -\frac{\bar{x}_i-\bar{x}_i'}{2\sqrt{n_2}}\right)$$

whence

$$\overline{OY}_i.\overline{OY}_j = \tfrac{1}{2}(\bar{x}_i-\bar{x}_i')(\bar{x}_j-\bar{x}_j'). \tag{24}$$

Also get the resultant of $\overline{OZ}_i$ and $\overline{OZ}_i'$ and denote it by $\overline{OV}_i(i=1, 2, \ldots, p)$ whence we obtain

$$\overline{OV}_i.\overline{OV}_j = n_1 c_{ij}. \tag{25}$$

Finally find the resultant of $\overline{OV}_i$ and $\sqrt{\left(\frac{2n_1n_2}{n_1+n_2}\right)}\overline{OY}_i$ and denote it by $\overline{OT}_i(i=1, 2, \ldots, p)$, then

$$\overline{OT}_i.\overline{OT}_j = a_{ij}. \tag{26}$$

Let us denote the foot of the perpendicular from A to the space formed by $\overline{OT}_1, \overline{OT}_2, \ldots, \overline{OT}_p$ by L. Denote the angle between $\overline{OL}$ and $\overline{OA}$ by ϕ and that between $\overline{OL}$ and $\overline{OT}_i$ by $\psi_i(i=1, 2, \ldots, p)$.

Now make the linear transformation and denote the corresponding points and vectors in the image space by the same symbols. Then it is easily seen that

$$T^2 = \sum(\bar{y}_i-\bar{y}_i')^2 = \cos^2\phi\{\sum_i \overline{OT}_i^2\cos^2\psi_i\}.$$

Since we have to build up the region out of (21), it comes out to be given by

$$\cos^2\phi \geq k \tag{27}$$

and since $\cos^2\phi$ is invariant under all linear transformation and is equal to D_1^2, the proof is complete.

The property noted above makes this test, as a particular case, uniformly most powerful among those tests which involve the single parameter Δ^2 in their power functions (Simaika, 1942).

5. *The parameter Δ^2 as a measure of divergence.* Bhattacharya (1943) defines a measure of divergence between two populations when the variates in one are in one-to-one correspondence with those of the other and the parameter Δ^2 comes out as such a measure of divergence.

This measure has got many elegant properties but it does not necessarily come into the power function of a test of the corresponding null hypothesis. As everywhere one has to proceed to measure divergence after applying tests of significance, it suggests itself

that the power function which measures the detectability of departures from the null hypothesis should be taken into account in defining a measure of divergence.

So a suitable test has got to be found out in the first place for the null hypothesis under consideration. The equi-detectable contour as function of the parameters independent of the statistic defines a measure of divergence. In this way also Δ^2 comes out as a measure of divergence if D^2-statistic having the optimum properties noted above is used for test of significance.

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ON THE RELATIVISTIC ANALOGUE OF EARNSHAW'S THEOREM ON THE STABILITY OF A PARTICLE IN A GRAVITATIONAL FIELD

By

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Introduction

In the Newtonian theory of gravitation the equilibrium of a particle in a gravitational field cannot be stable. This is known as Earnshaw's theorem. The theorem applies also to an electrostatic field. Strictly speaking such equilibrium may be stable for certain displacements, but there would always exist displacements for which it is unstable. The enunciation of this theorem in general relativity is beset with difficulties. For if the particle be at rest in one system of coordinates it will be in motion in another system. Equilibrium in the theory of relativity is as much a property of the coordinate system as of the state of motion of the particle. Nevertheless it is pertinent to investigate if the content of Earnshaw's theorem can be enunciated in a suitable form. We shall consider here only a static gravitational field. In such a field a definition of equilibrium is more natural and easier than in a general field. We shall show below that the proper coordinate system in which Earnshaw's theorem can be enunciated in a static gravitational field is one which is locally Galilean, as one can really expect. Gravitational phenomena described in such a system of coordinates will agree with the findings of a local observer in his immediate neighbourhood using his proper time and proper distance.

Introduction of locally Galilean coordinates on the geodesic of the particle at rest

We define the static gravitational field by the line element

$$ds^2 = g_{ik}dx_i dx_k + g_{44}dt^2$$
$$= -\bar{g}_{ik}dx_i dx_k + g_{44}dt^2, \quad (i,\, k = 1,\, 2,\, 3), \tag{1}$$

the coordinate-system being so chosen that $g_{14} = g_{24} = g_{34} = 0$ and the metric tensor is independent of t.

We consider a particle to be at rest in this gravitational field at the point P with coordinates $x_1 = a$, $x_2 = b$, $x_3 = c$. The four-dimensional geodesic B of this particle is a line through (a, b, c) coinciding with the time-axis. For this particle at rest in space

$$\left(\frac{dt}{ds}\right)^2 = \frac{1}{g_{44}},$$

g_{44} being a function of a, b, c only is constant on this geodesic. It will be no loss of generality to assume that on this geodesic $g_{44} = 1$, and so $dt/ds = 1$.

At any point P on this geodesic we introduce locally Cartesian coordinates in space so that $\bar{g}_{ik} = \delta_{ik}$ and $\partial\bar{g}_{ik}/\partial x_r$ $(i, k, r = 1, 2, 3)$ all vanish. Since

$$\frac{dx_i}{ds} = \frac{d^2x_i}{ds^2} = 0, \quad (i = 1, 2, 3)$$

on the geodesic B, its differential equations reduce to

$$\frac{d^2t}{ds^2} + \{44, 4\}\left(\frac{dt}{ds}\right)^2 = 0,$$

$$\{44, i\}\left(\frac{dt}{ds}\right)^2 = 0, \quad (i = 1, 2, 3)$$

whence

$$\{44, i\} = 0, \quad (i = 1, 2, 3) \tag{2}$$

the first equation being satisfied identically.

From equation (2) we obtain

$$\frac{\partial g_{44}}{\partial x_i} = 0, \quad (i = 1, 2, 3). \tag{3}$$

Thus we find that on introducing locally Cartesian coordinates in space at any point on the four-dimensional geodesic of the particle at rest we obtain a coordinate system which is locally Galilean all along the geodesic. It may be noted that equations (3) are invariant with respect to arbitrary transformation of space coordinates, since $\partial g_{44}/\partial x_i$ behave like the components of a covariant vector for such transformations.

Conversely, it can be easily shewn that if the line element of a static gravitational field be given in the form (1) and if at a point in space $\partial g_{44}/\partial x_i$, $(i = 1, 2, 3)$, vanish then that point must be a point of equilibrium.

Some properties of the Riemann-Christoffel tensor on the geodesic of the particle at rest

We next show that in this coordinate system a number of components of the Riemann-Christoffel tensor (briefly *R-C* tensor) vanishes on B. Since Christoffel's symbols all vanish along B this tensor can now be written as

$$B_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 g_{jl}}{\partial x_i \partial x_k}\right).$$

(i). If none of the suffixes i, j, k, l is 4, then

$$B_{ijkl} = -\bar{B}_{ijkl}, \quad (i, j, k, l = 1, 2, 3), \tag{4}$$

(ii) If only one suffix is 4, then

$$B_{4jkl} = \frac{1}{2}\left(\frac{\partial^2 g_{kl}}{\partial x_4 \partial x_j} - \frac{\partial^2 g_{jl}}{\partial x_4 \partial x_k}\right) = \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jl}}{\partial x_k}\right) = 0.$$

All such symbols are, therefore, equal to zero.

(iii) If only two of the suffixes are 4, then

$$B_{44kl} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_k \partial x_l}. \tag{5}$$

Hence the components which do not necessarily vanish are

$$B_{44kl}, \quad B_{4j4l}, \quad B_{i4k4}, \quad B_{ij44}.$$

(iv) Components with three suffixes as 4, of course, all vanish.

It is, however, possible to make those of the components B_{44kl} etc. mentioned in (iii) for which $k \neq l$ vanish by an orthogonal transformation of space-coordinates. For linear transformations of space coordinates the components B_{44kl}, $(k, l, = 1, 2, 3)$ will be transformed as a symmetrical tensor of the second rank. Hence it will be possible by an orthogonal transformation to pass on to a coordinate system in which B_{44kl}, vanishes when $k \neq l$. Moreover, it is easily seen that any such linear transformation of space coordinates leaves the components g_{i4} $(i = 1, 2, 3, 4)$ as well as the locally Galilean character of the coordinates along B unaltered. Thus we can make use of a coordinate system in which

$$B_{44kl} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_k \partial x_l} = 0, \text{ for } k \neq l;\ k, l = 1, 2, 3. \tag{6}$$

Hence in this coordinate system the only independent components of the R-C tensor with the suffix 4 which may be different from zero are

$$B_{4411}, \quad B_{4422}, \quad B_{4433}$$

All these reductions have been made simply by suitable choice of our coordinate system.

Case of empty space

We will have further reductions in the number of non-zero components of the R-C tensor on account of the properties of the gravitational field in empty space. Taking the field equations for empty space in the form

$$G_{ik} = 0$$

i.e., neglecting the cosmological term λ, and expressing the Einstein tensor G_{ik} in terms of the R-C tensor, we obtain the following relations at any point P on the geodesic B:

$$\left.\begin{aligned} 0 = G_{11} &= g^{ik}B_{11ik} = -B_{1122} - B_{1133} + B_{1144} \\ 0 = G_{22} &= g^{ik}B_{22ik} = -B_{2233} - B_{2211} + B_{2244} \\ 0 = G_{33} &= g^{ik}B_{33ik} = -B_{3311} - B_{3322} + B_{3344} \end{aligned}\right\} \tag{7}$$

$$0 = G_{44} = g^{ik}B_{44ik} = -B_{4411} - B_{4422} - B_{4433} \tag{8}$$

$$\left.\begin{aligned} 0 = G_{12} = g^{ik}B_{12ik} = -B_{1233}+B_{1244} \\ 0 = G_{23} = g^{ik}B_{23ik} = -B_{2311}+B_{2344} \\ 0 = G_{31} = g^{ik}B_{31ik} = -B_{3122}+B_{3144} \end{aligned}\right\} \tag{9}$$

From equations (7) and (8) we obtain the important relation

$$B_{1122}+B_{2233}+B_{3311} = 0. \tag{10}$$

Using this relation equations (7) can be written as

$$\left.\begin{aligned} B_{4411}+B_{3233} = 0, \\ B_{4422}+B_{3311} = 0, \\ B_{4433}+B_{1122} = 0. \end{aligned}\right\} \tag{11}$$

All these relations hold even if we do not use the coordinate system in which equations (6) hold. In the latter coordinates, however, we have, from equations (9),

$$B_{3312} = B_{1123} = B_{2231} = 0. \tag{12}$$

From equations (6), (11), and (12) we see that in the coordinate system we have used the only independent components of the R-C tensor which may have non-zero values are

$$B_{1122},\quad B_{2233},\quad B_{3311},\quad \text{and}\quad B_{4411},\quad B_{4422},\quad B_{4433}$$

and they are connected together by equations (10) and (11).

We can express the three components in equation (10) in terms of curvature in space. In an n-dimensional space if u^i and v^i are two unit vectors defining two directions at a point P then the Gaussian curvature at P of the geodesic surface determined by u^i and v^i is given by the formula

$$K = \frac{1}{\sin^2\alpha}\cdot B_{ijkl}\,u^i v^j u^k v^l, \tag{13}$$

where α is the angle between the vectors u^i and v^i. In our three-space if the coordinate axes be so chosen as to coincide with the principal directions of curvature then we shall have

$$\overline{B}_{1123} = \overline{B}_{2231} = \overline{B}_{3312} = 0$$

and, therefore,

$$B_{1123} = B_{2231} = B_{3312} = 0.$$

Equations (9) then give

$$B_{44ik} = 0, \quad \text{for} \quad i \neq k.$$

From this we see that the above choice of the coordinate-axes in space is equivalent to the orthogonal transformation adopted before.

The principal curvatures are, by (13),

$$K_1 = \overline{B}_{2323},\quad K_2 = \overline{B}_{3131},\quad K_3 = \overline{B}_{1212}.$$

On account of the equations (4) we shall have

$$K_1 = B_{2233},\quad K_2 = B_{3311},\quad K_3 = B_{1122} \tag{14}$$

The R-C symbols in the above equations correspond to the four-space.

EQUATIONS OF NORMAL DEVIATION OF THE NEIGHBOURING GEODESIC

In the Newtonian theory equilibrium is said to be stable if for any small displacement from the point of equilibrium the subsequent motion takes place in the immediate neighbourhood of the point. In the relativistic field if the particle be displaced from P to a point P' it will describe a geodesic passing through P'. The question of stability, therefore, leads to the geometrical problem of the behaviour of a geodesic infinitely near a given geodesic, or, of geodesic deviation, as it has been called by Levi-Civita (1927). Stability is ensured if the geodesic through P' remains confined to the immediate neighbourhood of the geodesic described by the particle at rest at P. The particle will be supposed to be initially at rest at P'. In this connection we shall make use of the following theorem due to Levi-Civita.

In a space of n dimensions let B be a given geodesic, and L another geodesic lying in its immediate neighbourhood. Let the coordinate system be so chosen as to be locally Cartesian at all points of B (Fermi, 1922). Such a coordinate system may be made to possess the following properties. "x_n is the arc σ of B measured from an arbitrary origin P upto a point Q on B; the x_i's ($i = 1, 2, \ldots n-1$) may be regarded as the components of the elementary vector QM (Q being the orthogonal projection upon B of any point M in its immediate neighbourhood) in $n-1$ directions mutually perpendicular and all perpendicular to B chosen arbitrarily at P and carried by parallelism along B." In these coordinates if x_i be a point on L then the behaviour of L is given by the following equations

$$\frac{d^2x_i}{d\sigma^2} = -\sum_{1}^{n-1}{}_a B^i{}_{nan}x_a, \quad i = 1, 2, \ldots, (n-1). \tag{15}$$

Now

$$B^i{}_{nan} = g^{ib}B_{nanb} = g^{ii}B_{nani}, \quad \text{(not summed)}.$$

In Einsteinian four-space the formulae, therefore, become

$$\frac{d^2x_i}{dt^2} = \sum_{1}^{3}{}_a B_{4a4i}x_a = -\sum_{1}^{3}{}_a B_{44ai}x_a, \quad i = 1, 2, 3. \tag{16}$$

If we substitute the values of B_{44ai} from equations (5) the formulae become

$$\frac{d^2x_i}{dt^2} = -\frac{1}{2}\sum_{1}^{3}{}_a \frac{\partial^2 g_{44}}{\partial x_a \partial x_i}x_a, \quad i = 1, 2, 3. \tag{17}$$

In the final coordinates used in equations (6) they reduce to

$$\frac{d^2x_i}{dt^2} = -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_i^2}x_i, \quad \text{(not summed)}, \quad i = 1, 2, 3; \tag{18}$$

or

$$\frac{d^2x_i}{dt^2} = -B_{44ii}x_i, \quad \text{(not summed)}, \quad i = 1, 2, 3, \tag{19}$$

with the condition (8), namely

$$B_{4411}+B_{4422}+B_{4433} = 0.$$

In view of equations (10), (11), and (14) the formulae (19) can be written in the equivalent form

$$\frac{d^2x_i}{dt^2} = K_i x_i, \quad \text{(not summed)}, \quad i = 1, 2, 3, \tag{20}$$

with the condition

$$K_1 + K_2 + K_3 = 0. \tag{21}$$

EARNSHAW'S THEOREM

From equations (20) and (21) we deduce Earnshaw's theorem. For in order that equilibrium may be stable at P the x_i's must remain permanently small, the condition for which is that K_1, K_2, K_3 must be all negative. But if they are all negative their sum will not be zero as demanded by equation (21). If the three quantities are all zero then we see from equations (11) and (14) that all the components of the R-C tensor vanish, and, therefore, space-time is locally flat in the neighbourhood of P. Equilibrium at P is then neutral to the first order. The outcome of all these discussions is that equilibrium of a particle in a static gravitational field may be either neutral to the first order or unstable, but can never be stable.

It should be noted that equations (18) and (19) are more general than equations (20). Equations (20) hold only when we take the field equations in empty space in the form $G_{ik} = 0$ neglecting the cosmological term λ. In discussing the question of stability of a particle in the cosmological models the formulae can be used only in the form (18) or (19) but not in the form (20). For example, in the Einstein Universe equilibrium is neutral at every point though space is curved.

Equations (20) and (21) imply that at every point P of equilibrium in a static gravitational field there are at least three mutually perpendicular directions for a displacement along which the force will be directed either towards or away from P. These are the three principal directions of curvature in space. And, unless space-time is locally flat in the neighbourhood of P, there will be at least one direction in which the infinitesimal motion will be simple harmonic, and at least one direction for which the force will be directed away from P. If two of the quantities K_1, K_2, K_3 are equal then for all directions on the geodesic surface determined by the corresponding principal directions the force per unit displacement from P will be the same and will be all directed either towards or away from P. In empty space the three quantities K_1, K_2, K_3 cannot be equal unless they are all zero (e.g., at the centre of a uniform spherical shell). Information about curvature in space-time is thus obtained from the irreducible gravitational field that remains in the neighbourhood of a point even after the introduction of locally Galilean coordinates at that point. In fact equations (20) provide us with a means of measuring the curvature of space near a point of equilibrium.

All these results bear a perfect resemblence to electrostatic phenomena in the neighbourhood of a point of equilibrium, the component g_{44} playing the part of the electrostatic potential φ. For, according to (8),

$$0 = B_{4411} + B_{4422} + B_{4433} = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) g_{44}$$

and, according to (3),

$$\frac{\partial g_{44}}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

Again in the neighbourhood of a point of equilibrium in an electrostatic field

$$\varphi = \varphi_0 + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} x_i x_k + \text{terms of higher order},$$

whence the force

$$F_i = -\frac{\partial^2 \varphi}{\partial x_a \partial x_i} x_a$$

neglecting terms of higher order. These expressions for the forces are perfectly analogous to those in equations (17), and in combination with Laplace's equation they lead to Earnshaw's theorem in electrostatics.

The case when space is not empty

If $\varrho \neq 0$ at P, then taking the field equations in the form

$$G_i^k - \tfrac{1}{2} g_i^k G = -8\pi T_i^k$$

and carrying out the calculations we get the following relations at P:

$$B_{1122} + B_{2233} + B_{3311} = 8\pi\varrho$$

Therefore, by (14),

$$K_1 + K_2 + K_3 = 8\pi\varrho > 0$$

and

$$\tfrac{1}{2}\nabla^2 g_{44} = 4\pi(\varrho + P_{xx} + P_{yy} + P_{zz})$$

where P_{xx} etc. are the stresses. This differs from Poisson's equation by the additional terms $4\pi(P_{xx}+P_{yy}+P_{zz})$. Comparing with equations (18) we see that, since $\varrho > 0$, it is possible to have stable equilibrium when space is not empty at P. Also,

$$B_{3312} - B_{4412} = 8\pi P_{xy},$$
$$B_{1123} - B_{4423} = 8\pi P_{yz},$$
$$B_{2231} - B_{4431} = 8\pi P_{zx}.$$

Directions of principal curvature in space, therefore, may not coincide with the principal directions of stress.

Stability of a particle in the proposed cosmological models

Since all points in the proposed models are in a certain sense equivalent it is sufficient to discuss stability at the origin of space coordinates.

(a) *Einstein Model.* Taking the line element in the isotropic form

$$ds^2 = -\frac{1}{[1+r^2/4R^2]^2}\cdot(dx_1^2 + dx_2^2 + dx_3^2) + dt^2$$

we see that the first derivatives of the g_{ik} all vanish at the origin and, therefore, along the geodesic $x_1 = x_2 = x_3 = 0$. Hence the coordinates are locally Galilean along the

geodesic. Further, since g_{44} is a constant, we see from equations (18) that the origin is a point of neutral equilibrium.

(b) *de Sitter Model.* We take the line element in the isotropic form

$$ds^2 = -\frac{1}{[1+r^2/4R^2]^2} \cdot (dx_1^2 + dx_2^2 + dx_3^2) + \left(1 - \frac{r^2}{R^2[1+r^2/4R^2]^2}\right) \cdot dt^2.$$

All the first derivatives of the g_{ik} vanish at the origin. Also

$$\frac{\partial^2 g_{44}}{\partial x_i \partial x_k} = 0, \text{ for } i \neq k, \text{ and } \frac{\partial^2 g_{44}}{\partial x_i^2} = -\frac{2}{R^2} \text{ for } i = 1, 2, 3.$$

Hence from equations (18) there is a force of repulsion proportional to distance from the origin, the force per unit displacement being $1/R^2$.

These well-known results enable us to verify our calculations.

The question of stability of a particle in the non-static models has also been investigated and will form the subject-matter of another paper.

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RADIAL OSCILLATIONS OF A ROTATING STAR

By

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1. Cepheids, in general, possess rotation and consequently do not preserve a strictly spherical shape. With a view to study the effect of rotation on the period of pulsation, I considered the radial oscillations of a slowly rotating homogeneous star elsewhere (1940). In the present note, I have considered the radial oscillations of the Stellar model in which the strata of equal density are similar spheroids and which is uniformly rotating so that at any given instant the angular velocity is same throughout the mass.

2. Taking the axis of rotation as z-axis and neglecting the square of radial velocity, the equations* of motion in the polar coordinates are:

$$\frac{d^2r}{dt^2} = \frac{\partial V}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + r\omega^2 \sin^2\theta, \tag{1}$$

$$0 = \frac{\partial V}{\partial \theta} - \frac{1}{\rho}\frac{\partial p}{\partial \theta} + r^2\omega^2 \sin\theta\cos\theta, \tag{2}$$

$$0 = \frac{1}{\omega}\frac{d\omega}{dt} + \frac{2}{r}\frac{dr}{dt}, \tag{3}$$

where p, ρ, V are the pressure, density and potential at any instant t at a point (r, θ, ϕ) and ω is the angular velocity. The suffix zero will denote the undisturbed values of these variables. Let

$$\left.\begin{aligned} r &= r_0(1+r_1), \quad & p &= p_0(1+p_1), \\ \rho &= \rho_0(1+\rho_1), \quad & \omega &= \omega_0(1+\omega_1). \end{aligned}\right\} \tag{4}$$

Then the equation (3) gives

$$\omega_1 = -2r_1. \tag{5}$$

If, at any instant, the angular velocity is same throughout the mass, ω_1 and hence r_1 is independent of spatial coordinates.

If the oscillations are adiabatic

$$p_1 = \gamma\rho_1, \tag{6}$$

where γ is the ratio of specific heats.

* These equations are easily deduced from the general hydrodynamical equations on putting

$$q_r = \frac{dr}{dt}, \quad q_\theta = 0, \quad q_\phi = r\sin\theta\,\omega; \quad \frac{\partial V}{\partial \phi} = \frac{\partial p}{\partial \phi} = \frac{\partial q_r}{\partial \phi} = 0.$$

Let a and $a(1-\epsilon)$ be the semi-major and semi-minor axes of the meridian section of the spheroid, where ϵ is the ellipticity. Neglecting the square of the ellipticity, the polar equation of the surface is

$$r = a(1-\epsilon\cos^2\theta), \tag{7}$$

the approximation being valid as, according to Clairaut's theorem on rotating bodies,

$$\epsilon \propto \omega^2/(2\pi G\bar{\rho}),$$

where $\bar{\rho}$ is the mean density.

Let a_0, a_0, $a_0(1-\epsilon)$ be the the semi-axes of the spheroid passing through the point (r_0, θ, ϕ) and let these expand to a, a, $a(1-\epsilon)$ respectively. The volume of the thin spheroidal shell of semi-axes a, a, $a(1-\epsilon)$ is $4\pi\lambda a^3(1-\epsilon)$, where

$$\lambda = \frac{da}{a} = \frac{dr}{r}.$$

Hence, if ρ denote the density of the spheroidal shell and ρ_0 its undisturbed value, from the conservation of mass, we have

$$\rho_0 a_0^3(1-\epsilon)\frac{dr_0}{r_0} = \rho a^3(1-\epsilon)\frac{dr}{r},$$

which with the help of (7) reduces to

$$\rho_0 r_0^2\, dr_0 = \rho\, r^2\, dr \tag{8}$$

or

$$\rho/\rho_0 = (1-3r_1).$$

Hence

$$\rho_1 = -3r_1 \quad \text{and} \quad p_1 = -3\gamma r_1. \tag{9}$$

The potential V of the spheroid at an external point (r, θ, ϕ) is given by

$$V = \frac{GM}{r} + \frac{A^3}{r^3}\left(\frac{1}{2}\omega^2 A^2 - \frac{GM\epsilon}{A}\right)\left(\cos^2\theta - \frac{1}{3}\right), \tag{10}$$

where M is the mass and A, $A(1-\epsilon)$ are semi-major and semi-minor axes of the surface of the spheroid. As the attraction of a spheroidal shell vanishes at an internal point, the value of the potential at an internal point (r, θ, ϕ) is given by

$$V = \text{const.} + \frac{GM(r)}{r} + \frac{a^3}{r^3}\left(\frac{1}{2}\omega^2 a^2 - \frac{GM(r)\epsilon}{a}\right)\left(\cos^2\theta - \frac{1}{3}\right),$$

where $M(r)$ is the mass of the spheroid through (r, θ, ϕ). The value of gravity at (r, θ, ϕ) is given by

$$g = -\frac{\partial V}{\partial r} = \frac{GM(r)}{r^2} + \frac{3a^3}{r^4}\left(\frac{1}{2}\omega^2 a^2 - \frac{GM(r)\epsilon}{a}\right)\left(\cos^2\theta - \frac{1}{3}\right)$$

$$= \frac{GM(r)}{a^2}\left[1 + \epsilon\sin^2\theta + \frac{(3\cos^2\theta - 1)\omega^2 a^3}{2GM(r)}\right] \tag{11}$$

and

$$g_0 = \frac{GM(r_0)}{a_0^2}\left[1 + \epsilon\sin^2\theta + \frac{3(\cos^2\theta - 1)\omega_0^2 a_0^3}{2GM(r_0)}\right], \tag{12}$$

where $M(r) = M(r_0)$ due to conservation of mass.

From (11) and (12) we get

$$\frac{g}{g_0} = \frac{a_0^2}{a^2}\left[1+\frac{3\cos^2\theta-1}{2GM(r_0)}\omega_0^2 a_0^3\left\{\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{a}{a_0}\right)^3-1\right\}\right]$$

$$= 1-r_1\left(2+\omega_0^2 r_0^3\,\frac{3\cos^2\theta-2}{2GM(r_0)}\right). \tag{13}$$

Using (4), (5), (9) and (13), the equation (1) reduces to

$$\frac{d^2 r_1}{dt^2} = -r_1\left[\frac{g_0}{r_0}\left\{(3\gamma-4)-\omega_0^2 r_0^3\frac{3\cos^2\theta-1}{2GM(r_0)}\right\}+(5-3\gamma)\omega_0^2\sin^2\theta\right] \tag{14}$$

and

$$0 = \omega_0^2 r_0\sin^2\theta-g_0-\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0}, \tag{15}$$

which is the equation of relative equilibrium in the undisturbed state.

Equation (14) gives the frequency σ of oscillation:

$$\sigma^2 = \frac{g_0}{r_0}\left[(3\gamma-4)-\omega_0^2 r_0^3\,\frac{3\cos^2\theta-1}{2GM(r_0)}\right]+(5-3\gamma)\omega_0^2\sin^2\theta$$

$$= \frac{4\pi}{3}(3\gamma-4)G\bar{\rho}_0+(5-3\gamma)\,\frac{3-5\cos^2\theta}{2}\omega_0^2, \tag{16}$$

since the mean density $\bar{\rho}_0$ of the spheroid passing through (r_0, θ, ϕ) is given by

$$\bar{\rho}_0 = \frac{3M(r_0)}{4\pi r_0^3}\left[1-\epsilon(3\cos^2\theta-1)\right].$$

Taking the average value of σ^2 over the surface of the spheroid, we get

$$\sigma^2_{\text{average}} = \frac{4\pi}{3}(3\gamma-4)G\bar{\rho}_0+\frac{2}{3}(5-3\gamma)\omega_0^2. \tag{17}$$

3. It is evident from (16) that the frequency σ depends on $\bar{\rho}_0$ and θ. In order that the period may be same throughout along a radius vector the star should be homogeneous or possibly the variation in its density is small being of the order of fourth or higher power of the angular velocity. To get the qualitative idea of the effect of rotation on the period of pulsation, the average value of the frequency is obtained in (17). From (17) we may conclude that the rotation decreases the period of uniform radial oscillation of the model if $\gamma < \frac{5}{3}$. It is interesting to note that an expression similar to (17) has been obtained by Ledoux (1945) by a different method but from his treatment one cannot fully realize the physical nature of the result.

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ON SOME PROPERTIES OF COMPLEX OPERATIONAL MATRICES

By

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The operational matrices treated in this paper are those derived from one of the special type

$$D_x = \sum_{j=1}^{n} e_{j+1,1}\,\xi_j, \tag{1}$$

where the e's are matrix units and ξ_j denotes the linear differential operator

$$\xi_j = \frac{1}{2}\left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right), \quad i = \sqrt{(-1)}. \tag{2}$$

The symbolic Hamiltonian operator ∇ in a real space of three or higher dimensions (Wedderburn, 1934) is evidently an analogue to it and instances occur in this paper where a known result is generalized for a complex space of n dimensions. The n-dimensional complex space is here conceived as one *accommodating* $2n$ co-ordinate vectors, half of which are real and the rest imaginary, so that an arbitrary vector in this space may be defined by means of $2n$ *real* co-ordinates.

Associated with (1) we have the matrix

$$X = \sum_{j=1}^{n} e_{j+1,1}\,x_j, \quad x_j = q_j + ip_j, \tag{3}$$

satisfying the relations

$$\begin{aligned} D_x X^\dagger &= \sum_{j=1}^{n} e_{j+1,j+1} = U_1, \\ \bar{D}_x X^\dagger &= 0, \end{aligned} \tag{4}$$

where the bar indicates simply the complex conjugate and the $\dagger$ the transposed with complex conjugate elements, so that the use of both the signs will indicate the transposed only.

The present paper is concerned chiefly with some typical formulæ indicating equivalence of matrix operations in a complex space. The laws which govern such operations are characteristic of the way in which the matrix units are introduced in (1) or (3). Based on a particular set of formulæ, a matrix representation of the conditions for a contact-transformation is also included in this paper.

Let dX denote a differential of the matrix (3), then it is easy to verify the equivalence of the operational matrices

$$dX^\dagger D_x + d\bar{X}^\dagger \bar{D}_x = e_{11} d, \tag{5}$$

where d stands for the operator

$$\sum_{k=1}^{n}\left(dq_k\frac{\partial}{\partial q_k} + dp_k\frac{\partial}{\partial p_k}\right).$$

Consider now the matrix

$$A = \sum_{j=1}^{n} e_{j+1,1}\, a_j, \quad a_j = m_j + il_j, \tag{6}$$

where m_j and l_j are single-valued real functions of $2n$ variables (q, p), continuous and differentiable partially with respect to q's and p's. We shall call A as a *function* of X.

Applying (5) to $A^\dagger$ we have for a linear displacement in complex space (Struik, 1934)

$$dA^\dagger = dX^\dagger(D_x A^\dagger) + d\overline{X}^\dagger(\overline{D}_x A^\dagger), \tag{7}$$

where omitting the summation signs, as being implied by the use of repeated suffixes

$$D_x A^\dagger = e_{j+1,k+1}\,\xi_j \bar{a}_k, \quad \overline{D}_x A^\dagger = e_{j+1,k+1}\,\bar{\xi}_j \bar{a}_k. \tag{8}$$

This summation-convention will be adopted in the rest of the paper.

It follows from (7) that higher order differentials of A, expressed in terms of those of X, contain the matrices (8) and those derived from them by the repeated application of the operational matrix of the diagonal type.

$$\Delta_1 = e_{j+1,j+1}\, d. \tag{9}$$

Let us next define a transformation, from the set of $2n$ variables (q, p) to another (v, u), by means of the $2n$ mutually independent equations

$$\left.\begin{aligned} q_j &= q_j(v_1, \ldots, v_n, u_1, \ldots, u_n), \\ p_j &= p_j(v_1, \ldots, v_n, u_1, \ldots, u_n), \end{aligned}\right\} \tag{10}$$

where the single-valued functions involved are real and continuous together with such of their derivatives as enter the discussion. It is understood that in the domain under consideration the transformation is reversible.

Introducing now the matrix

$$Y = e_{j+1,1}\, y_j, \quad y_j = v_j + iu_j \tag{11}$$

and the associated operational matrix

$$D_y = e_{j+1,1}\, \eta_j, \quad \eta_j = \frac{1}{2}\left(\frac{\partial}{\partial v_j} + i\frac{\partial}{\partial u_j}\right) \tag{12}$$

we can also express (7) in the form

$$dA^\dagger = dY^\dagger(D_y A^\dagger) + d\overline{Y}^\dagger(\overline{D}_y A^\dagger). \tag{13}$$

We have, in fact, the relations

$$\left.\begin{aligned} D_y A^\dagger &= (D_y X^\dagger)(D_x A^\dagger) + (D_y \overline{X}^\dagger)(\overline{D}_x A^\dagger) \\ \overline{D}_y A^\dagger &= (\overline{D}_y X^\dagger)(D_x A^\dagger) + (\overline{D}_y \overline{X}^\dagger)(\overline{D}_x A^\dagger) \end{aligned}\right\} \tag{14}$$

Putting $A = Y$ and using (4), we get from the above

$$\left.\begin{aligned} U_1 &= (D_y X^\dagger)(D_x Y^\dagger) + (D_y \overline{X}^\dagger)(\overline{D}_x Y^\dagger), \\ 0 &= (\overline{D}_y X^\dagger)(D_x Y^\dagger) + (\overline{D}_y \overline{X}^\dagger)(\overline{D}_x Y^\dagger). \end{aligned}\right\} \tag{15}$$

Further, the operational matrices D_x and D_y are connected by means of the equations

$$\left.\begin{aligned} D_y &= (D_yX^{\dagger})D_x+(D_y\overline{X}^{\dagger})\overline{D}_x,\\ \text{or}\qquad D_x &= (D_xY^{\dagger})D_y+(D_x\overline{Y}^{\dagger})\overline{D}_y. \end{aligned}\right\} \tag{16}$$

The equations (14), (15) and (16) are analogous to the corresponding formulæ of the Differential Calculus. By an extension of (5) to include differentials of several independent matrices of type X one obtains formulæ analogous to those involving partial derivatives.

3 The matrices (8) are of the general type

$$\Omega = e_{j+1,k+1}\,\omega_{jk}, \tag{17}$$

where ω's are functions of (q, p). To distinguish particular types, such as symmetric, skew-symmetric, Hermitian and anti-Hermitian, we shall make use of the symbols Ω_s, Ω_{sa}, Ω_h, Ω_{ha} respectively,

Since $A^{\dagger}\Omega$ is of type $A^{\dagger}$ we can apply D_x to it and the result is expressible in the form

$$D_x(A^{\dagger}\Omega) = (D_xA^{\dagger})\Omega+(AD_x^{\dagger})^{\dagger}\Omega. \tag{18}$$

where the new operational matrix involved is

$$(AD_x^{\dagger})^{\dagger} = e_{j+1,k+1}\,\bar{a}_k\xi_j. \tag{19}$$

As regards its properties, we note the following:

$$\left.\begin{aligned} (AD_x^{\dagger})^{\dagger}B &= (D_x\overline{B}^{\dagger})A = D_x(A^{\dagger}B)-(D_xA^{\dagger})B,\\ \{(AD_x^{\dagger})^{\dagger}\Omega\}B &= \{(\overline{B}D_x^{\dagger})^{\dagger}\,\overline{\Omega}^{\dagger}\}\overline{A},\\ \Omega(AD_x^{\dagger})^{\dagger} &= \{A(\Omega D_x)^{\dagger}\}^{\dagger},\\ dX^{\dagger}(AD_x^{\dagger})^{\dagger}\Omega+d\overline{X}^{\dagger}(A\overline{D}_x^{\dagger})^{\dagger}\Omega &= A^{\dagger}\Delta_1\Omega. \end{aligned}\right\} \tag{20}$$

We observe that (19) can operate on any matrix, either of type A or Ω, yielding a matrix of the same type, so that the operation may be repeated without any change in the operand.

When D_x-operates on a matrix of type e_{11}, the resulting matrix is of type A. Let A_0 denote the matrix

$$A_0 = e_{11}a_0, \quad a_0 = m_0+il_0, \tag{21}$$

where m_0, l_0 are functions of (q, p). Then a matrix A of the form $\overline{D}_xA_0$ satisfies the equation

$$(\overline{D}_x\overline{A}^{\dagger})^{\dagger}-(D_xA^{\dagger}) = 0. \tag{22}$$

It may be noted that

$$\overline{D}_x(A_0\overline{B}^{\dagger}) = (\overline{D}_xA_0)\overline{B}^{\dagger}+a_0(\overline{D}_x\overline{B}^{\dagger}), \tag{23}$$

where a_0 serves as a multiplier to every element in $\overline{D}_x\overline{B}^{\dagger}$.

4. The operational matix $\overline{D}_x^{\dagger}$ can operate on a matrix either of type A or of type Ω. If, in particular, A is of the form

$$A = (D_x^{\dagger}\Omega_{sa})^{\dagger} \tag{24}$$

it follows that

$$\overline{D}_x^{\dagger}A = 0. \tag{25}$$

Some of the properties of $\overline{D}_x^{\dagger}$ are noted below:

$$\left.\begin{aligned}\overline{D}_x^{\dagger}(BA_0) &= (\overline{D}_x^{\dagger}B)A_0+\overline{B}^{\dagger}(D_xA_0),\\ \overline{D}_x^{\dagger}(AB^{\dagger}) &= (\overline{D}_x^{\dagger}A)B^{\dagger}+\overline{A}^{\dagger}(D_xB^{\dagger}),\\ \overline{D}_x^{\dagger}(\Omega A) &= (\overline{D}_x^{\dagger}\Omega)A+(\Omega^{\dagger}\overline{D}_x)^{\dagger}A.\end{aligned}\right\} \tag{26}$$

The new operational matrix involved in the last of the above is

$$(\Omega^{\dagger}\overline{D}_x)^{\dagger} = e_{1,k+1}\,\omega_{jk}\,\xi_j, \tag{27}$$

which is connected with (19) by means of the third equation in the set (20). It may be noted that when the operand is A, the result of operating by (27) is simply obtained by multiplying each element of Ω by the corresponding element of $D_x\overline{A}^{\dagger}$. Let us consider next the evaluation of

$$(\Omega_{sa}^{\dagger}\overline{D}_x)^{\dagger}\overline{\Omega}_{ra}^{\dagger}, \tag{28}$$

which is equivalent to

$$e_{1,l+1}\,\omega_{jk}\,\xi_j\,\omega_{lk}. \tag{29}$$

If we write

$$\omega_{jk}\,\xi_j\,\omega_{lk} = \omega_{jk}[l,k,j]+\tfrac{1}{2}\xi_l(\omega_{jk}\,\omega_{jk}),$$

where

$$[l,k,j] = \tfrac{1}{2}(\xi_j\,\omega_{lk}+\xi_k\,\omega_{jl}+\xi_l\,\omega_{kj}),$$

then it follows that

$$(\Omega_{sa}^{\dagger}\overline{D}_x)^{\dagger}\overline{\Omega}_{sa}^{\dagger} = e_{1,l+1}\,\omega_{jk}[l,k,j]+\tfrac{1}{2}\overline{D}_x^{\dagger}(e_{l+1,l+1}\,\omega_{jk}\,\omega_{jk}). \tag{30}$$

If in particular

$$\Omega_{sa} = (\overline{D}_x\overline{A}^{\dagger})^{\dagger}-D_xA^{\dagger},$$

the expression $[l,k,j]$ vanishes and (30) is further simplified. We give below a set of formulæ involving successive partial operations:

$$\left.\begin{aligned}D_x^{\dagger}(\overline{D}_y\overline{A}^{\dagger})^{\dagger} &= (\overline{D}_y\overline{D}_x^{\dagger}A)^{\dagger},\\ (AD_x^{\dagger})^{\dagger}D_y &= (\overline{A}^{\dagger}\overline{D}_yD_x^{\dagger})^{\dagger},\\ D_y^{\dagger}(AD_x^{\dagger})^{\dagger} &= (A\overline{D}_y^{\dagger}D_x)^{\dagger}+\{(\overline{D}_yA^{\dagger})^{\dagger}\overline{D}_x\}^{\dagger},\\ D_y^{\dagger}D_x(A^{\dagger}\Omega) &= (D_y^{\dagger}D_xA^{\dagger})\Omega+\{(D_xA^{\dagger})^{\dagger}D_y\}^{\dagger}\Omega+D_y^{\dagger}(AD_x^{\dagger})^{\dagger}\Omega,\\ (d\lambda D_y^{\dagger})^{\dagger}D_xA^{\dagger}+(d\overline{X}D_y^{\dagger})\overline{D}_xA^{\dagger} &= \Delta_1(D_yA^{\dagger}).\end{aligned}\right\} \tag{31}$$

5. Let us next pass on to the matrix treatment of the contact-transformation. We start with the definition that the transformation (10) represents a contact-transformation if

$$dY^{\dagger}\delta Y-d\overline{Y}^{\dagger}\delta\overline{Y} = dX^{\dagger}\delta X-d\overline{X}^{\dagger}\delta\overline{X}, \tag{32}$$

d and δ being two independent differential symbols. Expressing the above in terms of dX and δX by means of (7), the conditions for a contact-transformation can be represented in the form

$$\left.\begin{aligned}(D_xY^{\dagger})(D_xY^{\dagger})^{\dagger}-(D_x\overline{Y}^{\dagger})(D_x\overline{Y}^{\dagger})^{\dagger} &= U_1,\\ (D_xY^{\dagger})(\overline{D}_xY^{\dagger})^{\dagger}-(D_x\overline{Y}^{\dagger})(\overline{D}_x\overline{Y}^{\dagger})^{\dagger} &= 0.\end{aligned}\right\} \tag{33}$$

The above equations, by virtue of (15), lead to the conditions

$$\left.\begin{aligned}(D_xY^\dagger)^\dagger &= D_yX^\dagger,\\ (D_x\overline{Y}^\dagger)^\dagger &= -\overline{D}_yX^\dagger.\end{aligned}\right\} \tag{34}$$

An alternative form of (33) is then expressed as follows:

$$\left.\begin{aligned}(D_xY^\dagger)^\dagger(D_xY^\dagger)-(\overline{D}_xY^\dagger)^\dagger(\overline{D}_xY^\dagger) &= U_1,\\ (D_xY^\dagger)^\dagger(D_x\overline{Y}^\dagger)-(\overline{D}_xY^\dagger)^\dagger(\overline{D}_x\overline{Y}^\dagger) &= 0.\end{aligned}\right\} \tag{35}$$

Let us denote $(D_xY^\dagger)^\dagger$ and $(D_x\overline{Y}^\dagger)^\dagger$ by Φ and Ψ respectively, then

$$\left.\begin{aligned}\Phi^\dagger+\Psi^\dagger &= 2e_{j+1,k+1}\,\xi_j v_k, \quad \Phi^\dagger-\Psi^\dagger = -2ie_{j+1,k+1}\,\xi_j u_k\\ \Phi+\Psi &= 2e_{j+1,k+1}\,\overline{\xi}_k v_j, \quad \Phi-\Psi = 2ie_{j+1,k+1}\,\overline{\xi}_k u_j\end{aligned}\right\} \tag{36}$$

It may be seen that the conditions for a contact-transformation involving Poisson bracket expressions are given by the following:

$$\left.\begin{aligned}(\Phi-\Psi)(\Phi^\dagger-\Psi^\dagger)-\text{conj.} &= 0,\\ (\Phi+\Psi)(\Phi^\dagger-\Psi^\dagger)+\text{conj.} &= 2U_1,\\ (\Phi-\Psi)(\Phi^\dagger+\Psi^\dagger)+\text{conj.} &= 2U_1,\\ (\Phi+\Psi)(\Phi^\dagger+\Psi^\dagger)-\text{conj.} &= 0,\end{aligned}\right\} \tag{37}$$

which follow from (35).

The Lagrangian bracket expressions are, on the other hand, involved in conditions written in the form

$$\left.\begin{aligned}(\Phi^\dagger-\overline{\Psi}^\dagger)(\Phi-\overline{\Psi})-\text{conj.} &= 0,\\ (\Phi^\dagger+\overline{\Psi}^\dagger)(\Phi-\overline{\Psi})+\text{conj.} &= 2U_1,\\ (\Phi^\dagger-\overline{\Psi}^\dagger)(\Phi+\overline{\Psi})+\text{conj.} &= 2U_1,\\ (\Phi^\dagger+\overline{\Psi}^\dagger)(\Phi+\overline{\Psi})-\text{conj.} &= 0,\end{aligned}\right\} \tag{38}$$

which follow from (33).

A matrix method of obtaining further results in this connection may be of some interest.

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ON THE ENDOMORPHIC MAPPINGS $\{m\}$ OF A GROUP

By

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This paper is an outcome of a number of papers (see references at the end) written by Prof. F. W. Levi. Levi (1944a)* has shown that for every group there is a characteristic integral number M, and has also proved that the values of m for which the mapping $\{m\}$—i.e., the mapping $a \to a^m$ is an endomorphism for the group satisfy the relation $m(1-m) \equiv 0 \pmod M$, that is, they are idempotents of the residue classes mod M. The values of M and endomorphic $\{m\}$ in L-groups (Levi, 1942) and S-products (Levi, 1944b) of two groups are investigated in § 1 and § 2 respectively. In § 3 it is our aim to find out those values of m for which the mapping $\{m\}$ is an automorphism for the group.

§ 1. Values of M in L-groups

Let $\{m\}$ denote the mapping by which every element a of a group G is mapped on its mth power a^m, and let M be the smallest positive value of m such that $\{m\}$ maps G endomorphically on an Abelian subgroup. A group has been called an L-group if the associative law for the commutator-operation $((a, b), c) = (a, (b, c))$ is satisfied whenever two of the elements a, b, c are equal. We shall here find out the admissible values of M and endomorphic $\{m\}$ in groups of the above type. The results are given in the following propositions:

(I). *To every positive integer* $M_0 \not\equiv 2 \pmod 4$ *there exists an* L-*group for which* $M = M_0$.

(II). *For every* L-*group* $M \not\equiv 2 \pmod 4$.

(III). *For every* L-*group the endomorphic* $\{m\}$ *runs over all the idempotent residue classes* mod M.

Lemma: The above propositions hold for L-groups generated by two elements.

Proof of lemma: Let L_2 be an L-group generated by two elements, say a and b. We know that every element of L_2 can be represented by an ordered triad of integers with the rule of composition

$$SS' = [p+p',\ q+q';\ r+r'-p'q] \tag{1.1}$$

where $S = [p, q; r]$ and $S' = [p', q'; r']$ are any two elements of L_2. Hence

$$(SS')^k = [k(p+p'),\ k(q+q');\ k(r+r'-qp')-\tfrac{1}{2}k(k-1)(p+p')(q+q')]$$

and

$$S^kS'^k = [k(p+p'),\ k(q+q');\ k(r+r')-\tfrac{1}{2}k(k-1)(pq+p'q')-k^2qp'].$$

Now $\{k\}$ is an endomorphism if and only if

$$\begin{aligned} I = (SS')^k(S^kS'^k)^{-1} &= [0,\ 0;\ \tfrac{1}{2}k(k-1)(qp'-pq')] \\ &= [0,\ 0;\ qp'-pq']^{k(k-1)/2} = (S, S')^{k(k-1)/2}. \end{aligned} \tag{1.2}$$

* We have used M in place of m_0 in this paper.

Hence $\{k\}$ is an endomorphism if and only if $\mu | k(k-1)/2$ where μ is the order of the commutator (a, b) which generates the commutator-group of L_2. If the commutator-group is an infinite cyclic group, then $\mu = 0$* and the only possible endomorphic $\{m\}$ are $\{0\}$ and $\{1\}$; in this case $M = 0$. On the other hand, if the order of the commutator-group is finite, then $\mu > 0$. Now M must be a multiple of μ; again if $\{c\mu\}$ is an endomorphism where c is an integer, then $\{c\mu\}$ maps the group on an Abelian subgroup. Hence $M = c\mu$, where c is the smallest positive integer for which $\mu | c\mu(c\mu-1)/2$. Thus we have

$$\left.\begin{array}{ll} \text{case (1):} & \text{if } \mu \text{ is even, } M = 2\mu, \\ \text{case (2):} & \text{if } \mu \text{ is odd, } M = \mu. \end{array}\right\} \tag{1.3}$$

Hence M is either a multiple of 4 or an odd number, that is, $M \not\equiv 2 \pmod 4$. Given any positive integer n, we can construct an L_2 having its $\mu = n$ by simply taking the last coordinate r mod n in the ordered triad representation $[p, q; r]$ of the elements of L_2; hence it follows that we can construct an L_2 having its $M = M_0$, where M_0 is any arbitrary positive integer $\not\equiv 2 \pmod 4$.

If m is an idempotent of the residue classes mod M, that is, if $m(m-1) \equiv 0 \pmod M$ we have in case (1)

$$m(m-1) \equiv 0 \pmod{2\mu}$$

Therefore $\mu | m(m-1)/2$; hence $\{m\}$ is an endomorphism.

In case (2)

$$m(m-1) = \mu t$$

where t must be even. Therefore $\mu | m(m-1)/2$; hence $\{m\}$ is an endomorphism.

Thus the lemma has been established. The lemma implies Proposition (I).

Proof of the Propositions **(II)** *and* **(III)**: Let L be an L-group with $M \equiv 2 \pmod 4$. Put $M = 2d$, where d is odd. Any two elements a, b of L generate a subgroup S satisfying the conditions of the lemma. $\{2d\}$ is an endomorphism of S mapping it on an Abelian subgroup; hence the value of M for S is a factor of $2d$, and therefore a factor of d. Therefore $\{d\}$ maps (a, b) on I and ab on $a^d b^d$. As this holds for every pair of elements $a \in L$, $b \in L$, it follows that M is a factor of d, contrary to the supposition. Hence for any L-group $M \not\equiv 2 \pmod 4$.

Again let m be an idempotent of the residue classes mod M; then it is also an idempotent of the residue classes mod. any factor of M. Hence, in consequence of the lemma, $\{m\}$ is an endomorphism of the subgroup S. Therefore $\{m\}$ is also an endomorphism of L

All the propositions have thus been established.

A group whose commutator-group lies in the centre has been called an S-group (Levi, 1942). Hence every S-group is an L-group and an L-group generated by two elements is an S-group. Therefore the three propositions above are also true for S-groups.

* We adopt in this paper the convention of assigning the order zero to a group element which generates an infinite cyclic group.

§ 2. VALUES OF M FOR THE S-PRODUCT OF TWO GROUPS

Let A and B be two groups whose elements are denoted by the corresponding small letters. Then the elements of the S-product $A \circ B$ can be represented in the form $ab\gamma$, where γ is an element of $[A, B]$, with the rule of composition

$$(a_1 b_1 \gamma_1)(a_2 b_2 \gamma_2) = a_1 a_2 b_1 b_2 \gamma_1 \gamma_2 [a_2, b_1^{-1}]. \tag{2.1}$$

If $\{m\}$ is an endomorphism for $A \circ B$ it is necessarily an endomorphism for both A and B, but the converse is not true unless

$$\{(a_1 b_1 \gamma_1)(a_2 b_2 \gamma_2)\}^m = (a_1 b_1 \gamma_1)^m (a_2 b_2 \gamma_2)^m$$

always holds. From the rule of composition (2.1) we get

$$(ab\gamma)^m = a^m b^m \gamma^m [a, b^{-1}]^{m(m-1)/2}.$$

Hence the condition

$$\{(a_1 b_1 \gamma_1)(a_2 b_2 \gamma_2)\}^m = (a_1 b_1 \gamma_1)^m (a_2 b_2 \gamma_2)^m$$

gives after direct calculation the relation

$$[a_1, b_2]^{m(m-1)/2} = [a_2, b_1]^{m(m-1)/2}. \tag{2.2}$$

Since the a's and b's are arbitrary elements of A and B, it follows that a common endomorphism $\{m\}$ for A and B will be an endomorphism for $A \circ B$ if and only if $[a, b]^{m(m-1)/2} = I$, where a runs over A, and b over B.

Proposition (I): *When A and B are Abelian, the three propositions proved for L-groups hold true for the S-product $A \circ B$.*

Proof: When A and B are Abelian, the commutator-group of $A \circ B$ lies in the centre; hence $A \circ B$ is an S-group, and therefore the last two propositions proved for L-groups hold true. Again, given any $M_0 \not\equiv 2 \pmod 4$, we can obtain an S-product $A \circ B$ with $M = M_0$ by taking as A and B two different cyclic groups of order $M_0/2$ if M_0 is even, or of order M_0 if M_0 is odd.

The value of M for the direct product $A \times B$ can be at once calculated from the corresponding values for A and B, but no such calculation is possible for the S-product $A \circ B$ in the absence of further informations about A and B, for the value of M for $A \circ B$ depends upon the orders of the elements of the group $[A, B]$. Again $[A, B]$ is isomorphic to $[A/C(A), B/C(B)]$, and the orders of the elements of the latter depend upon the orders of the basis elements of $A/C(A)$ and $B/C(B)$. But the following simple example will show that the value of M for a group A does not in general put any restrictions on $A/C(A)$. Let

$$A = P \times G,$$

where P is a perfect group, that is, $P = C(P)$, and G is any arbitrary Abelian group.

Then the value of M for A is the same as that for P, but $P/C(P) \cong I$ where as

$$A/C(A) = P \times G/C(P \times G) = P \times G/C(P) \cong G.$$

§ 3. Automorphisms of the type $\{m\}$

Let the mapping $\{m\}$ give an automorphism for the non-Abelian group G.

Proposition (I): *$\{m\}$ is an automorphism only if $m \equiv 1 \pmod{M}$.*

Proof: If possible, let $m \not\equiv 1 \pmod{M}$ Then $\{1-m\}$ which is an endomorphism cannot map all elements of G on the centre, for $1-m$ is not a multiple of M; hence we can find two elements a and b such that $(a^{1-m}, b) = C \neq I$. But $(a^{1-m}, b)^m = (a^{m(1-m)}, b) = I$. Hence $\{m\}$ maps $C(\neq I)$ on I, and therefore it is not an automorphism.

The trivial case $m = 1$ shows that automorphisms $\{m\}$ actually exist. The condition $m \equiv 1 \pmod{M}$, that is, $m = kM+1$ is not sufficient for automorphism, because for some values of k, $kM+1$ may be divisible by the order of one or more elements of G.

When G is Abelian every $\{m\}$ is an endomorphism but it is an automorphism if and only if it is not divisible by the order of any element of G.

For any group G, $\{kM+1\}$ is always an endomorphism; it is an automorphism if and only if $a^{kM+1} \neq I$ for every $a \neq I$. If $a^{kM+1} = I$, then $a = (a^{-k})^M$, that is $a \in Z'$, where Z' denotes that subgroup of the centre on which the whole group is mapped by $\{M\}$. This shows that whether $\{kM+1\}$ is an automorphism or not depends only upon the orders of the elements of Z'. If Z' contains only the unit element I, then every $\{kM+1\}$ is the trivial automorphism $\{1\}$. From the value of M only it is not possible to decide whether a particular $\{kM+1\}$, $k \neq 0$, is an automorphism or not. For example, let some $\{kM+1\}$ be an automorphism for G. Consider the group $G \times A$ where A is a cyclic group of order $kM+1$. $G \times A$ has the same M as G, but $\{kM+1\}$ is not an automorphism for $G \times A$.

Proposition (II): *$\{kM+1\}$ is an automorphism for G if and only if it is an automorphism for Z'.*

Proof: By $\{kM+1\}$ an element $a^M \in Z'$ is mapped on $(a^M)^{kM+1} = (a^{kM+1})^M \in Z'$. Hence an automorphism $\{kM+1\}$ of G is an automorphism of Z'. On the other hand, we have already seen that if $a^{kM+1} = I$, then $a \in Z'$.

From the above proposition we immediately get the

Corollary: *If A is the subgroup of elements (of Z') mapped on I by $\{kM+1\}$ then $\{kM+1\}$ is an automorphism for G/A.*

Let q be the L.C.M. of the orders of the elements of Z'. We shall put $q = 0$ if Z' contains any element of order zero, or if there is no upper bound of the orders of the elements of Z'.

Proposition (III). *Two mappings $\{kM+1\}$ and $\{k'M+1\}$ are not different if and only if $k \equiv k' \pmod{q}$.*

Proof: If $\{kM+1\}$ and $\{k'M+1\}$ give the same mapping then $a^{kM+1} = a^{k'M+1}$ for every element $a \in G$, that is, $(a^M)^{k-k'} = I$. Therefore $c^{k-k'} = I$ for every element $c \in Z'$: hence $k-k' \equiv 0 \pmod{q}$. Conversely, let $k-k' \equiv 0 \pmod{q}$; then $(a^M)^{k-k'} = I$ for every $a \in G$. Therefore $a^{kM+1} = a^{k'M+1}$.

We now confine our investigation to those groups for which $q > 0$.

Proposition (IV). *$\{kM+1\}$ is an automorphism if and only if $(kM+1, q) = 1$.*

Proof: The order of any element of Z' is a factor of q, and corresponding to any factor of q there is at least one element in Z' whose order is equal to that factor.

Proposition (V). *In order that every $\{kM+1\}$ may be an automorphism it is necessary and sufficient that no element $\neq I$ of Z' has its order relatively prime to M.*

Proof: Suppose $(M, t) = 1$ where t is the order of some element $a \in Z'$. Then we can choose two integers f and g such that $fM - gt = -1$. Therefore $a^{fM+1} = a^{gt} = I$, showing thereby that there are values of k for which $\{kM+1\}$ is non-automorphic. Conversely, if every prime-factor of q is a factor of M, then $(kM+1, q) = 1$ for all values of k. Hence from Prop. (IV) it follows that every $\{kM+1\}$ is an automorphism.

The last proposition together with Prop. (III) gives the result that if every prime-factor of q is a divisor of M, then the total number of different automorphisms $\{m\}$ is exactly equal to q. This result is a particular case of the next Proposition

Let $q = q_1 q_2$, where q_2 is the greatest factor of q relatively prime to M, and $N =$ the total number of different automorphisms $\{m\}$ of G.

Proposition (VI). *$N = q_1\phi(q_2)$, where $\phi(q_2)$ denotes the number of integers not greater than and relatively prime to q_2.*

Proof: Let $q_2 = p_1^{i_1} p_2^{i_2} \ldots p_t^{i_t}$ where the p's are different prime numbers. The property $(kM+1, q) = 1$ can only be destroyed by $kM+1$ being divisible by one or more p's. Since p_j is prime and not a divisor of M, there is one and only one solution for $k (\text{mod } p_j)$ of the equation $kM \equiv -1 \pmod{p_j}$, that is, there is one and only one integer k in each set of p_j consecutive integers for which $kM+1$ is divisible by p_j. Hence, exactly q/p_j values of $k (\text{mod } q)$ become inadmissible on account of the prime-factor p_j. Taking care that a value of k which becomes inadmissible on account of more than one prime-factor is nevertheless counted only once we find that the total number of inadmissible values of $k \pmod q$ is equal to

$$\frac{q}{p_1} + \left(\frac{q}{p_2} - \frac{q}{p_1 p_2}\right) + \left(\frac{q}{p_3} - \frac{q}{p_1 p_3} - \frac{q}{p_2 p_3} + \frac{q}{p_1 p_2 p_3}\right) + \cdots = q \sum_{n=1}^{t} (-1)^{n-1} \frac{1}{p_{j_1} p_{j_2} \cdots p_{j_n}}, \quad (3.1)$$

where $(j_1, j_2, \ldots, j_n)$ runs over the different combinations of n integers out of the set $1, 2, \ldots, t$. Therefore

$$N = q\left[1 - \sum_{1}^{t} (-1)^{n-1} \frac{1}{p_{j_1} p_{j_2} \cdots p_{j_n}}\right] = q \prod_{r=1}^{t} \left(1 - \frac{1}{p_r}\right) = q_1 \phi(q_2).$$

Since

$$(a^{k_1 M+1})^{k_2 M+1} = a^{(k_1 k_2 M + k_1 + k_2)M+1}$$

the product of two automorphisms of the type $\{kM+1\}$ gives an automorphism of the same type. The commutative and the associative laws can be easily verified. Thus the automorphisms $\{kM+1\}$ form a commutative semigroup with $\{1\}$ as the unit element. Now $\{k_1 M+1\}$ has got an inverse $\{k_2 M+1\}$ provided there is some integer k_2 satisfying the relation

$$k_1 k_2 M + k_1 + k_2 \equiv 0 \pmod q,$$

or

$$k_2(k_1 M+1) \equiv -k_1 \pmod q.$$

Since $(k_1M+1,\ q)=1$, an integral solution for k_2 exists. Hence every element of the semigroup has an inverse; so the semigroup is a group. Thus we get

Proposition (VII). *The automorphisms $\{m\}$ of a group G form an Abelian group A.*

Noticing that in the product of two automorphisms the coefficients of M combine according to the rule

$$k_1 \circ k_2 = Mk_1k_2+k_1+k_2 \pmod{q},$$

we can obtain a true matrix representation of the Abelian group A by making $\{kM+1\}$ correspond to the matrix

$$\begin{pmatrix} kM+1 & k \\ 0 & 1 \end{pmatrix}$$

where k is taken mod q.

Every element of A induces an automorphism of Z', which in particular cases may be the identity-automorphism. Now $\{kM+1\}$ will induce the identity-automorphism of Z' if and only if $q \mid kM$, that is $q/(M,\ q) \mid k$. These automorphisms form a normal subgroup C of the group A. The order of C is evidently $(M,\ q)$. Hence

$$n_1 = \text{total number of different induced automorphisms of } Z'$$

$$= \text{index of } C \text{ in } A = \frac{N}{(M,\ q)} = \frac{q_1\phi(q_2)}{(M,\ q)}. \tag{3.2}$$

But since Z' is Abelian, the total number of different possible automorphisms $\{m\}$ of Z' is equal to

$$\phi(q) = \phi(q_1)\phi(q_2) = n_2, \text{ say.} \tag{3.3}$$

Hence $n_1 = n_2$ if and only if

$$\frac{q_1}{(M,\ q)} = \phi(q_1)$$

i.e.,

$$(M,\ q) = \frac{q_1}{\phi(q_1)} = \frac{d_1}{d_1-1}\cdot\frac{d_2}{d_2-1}\cdots\frac{d_n}{d_n-1} \tag{3.4}$$

where the d's are different prime numbers and $q_1 = \prod_{r=1}^{n} d_r^{i_r}$.

Except for the trivial case $q=1$, the right side of (3.4) can be an integer only when $q_1 = 2^s$ or 2^s3^t. Hence $(M,\ q) = 2$ or 3; but since every prime factor of q_1 is a divisor of M, it follows that $(M,\ q)=2$ and $q_1 = 2^s$. Conversely, if $(M,\ q)=2$, then $q_1 = 2^s$; therefore

$$\frac{q_1}{\phi(q_1)} = 2 = (M,\ q).$$

Hence we have

Proposition (VIII). *The necessary and sufficient condition that the automorphisms induced in Z' by the automorphisms $\{m\}$ of G are the only possible automorphisms $\{m\}$ of Z' is that $(M,\ q)=2$ when $q \neq 1$.*

C is a normal subgroup of A, but C may or may not be a direct factor of A. The following two examples illustrate both these possibilities.

Example 1. $M = 10$, $q = 12$.

The values of k (mod q) for which $\{kM+1\}$ are automorphic are

$$0, 1, 3, 4, 6, 7, 9, 10.$$

The values of k (mod q) for the elements of C are 0, 6.

Hence $C = (\{1\}, \{61\})$, and the four cosets of C in A are

$$(\{1\}, \{61\});\quad (\{11\}, \{71\});\quad (\{31\}, \{91\});\quad (\{41\}, \{101\}).$$

In this case it is possible to select one representative from each coset in such a way that the representatives form a subgroup. $(\{1\}, \{11\}, \{31\}, \{101\})$ is one such selection. Hence C is a direct factor and

$$A = (\{1\}, \{61\}) \times (\{1\}, \{11\}, \{31\}, \{101\}).$$

Example 2. $M = 6$, $q = 8$.

The values of k (mod q) for which $\{kM+1\}$ are automorphic are

$$0, 1, 2, 3, 4, 5, 6, 7.$$

The values of k (mod q) for the elements of C are 0, 4.

Hence $C = (\{1\}, \{25\})$, and the four cosets of C in A are

$$(\{1\}, \{25\});\quad (\{7\}, \{31\});\quad (\{13\}, \{37\});\quad (\{19\}, \{43\}).$$

Since $\{13\}^3 = \{37\}$ and $\{37\}^3 = \{13\}$, it is not possible to select one representative from each coset in such a way that the representatives form a subgroup. Hence C is not a direct factor of A.

None of the above cases can be ruled out by arguing that certain values of q may not be associated with certain values of M, for we can actually construct groups in which any given $M \neq 2$ is associated with any value of q. Take a group G which belongs* to M, that is, $\{M\}$ maps G on the unit element and there is no positive integer $m < M$ such that $\{m\}$ maps G endomorphically on an Abelian subgroup. Suppose we want to associate some given q with this M. Let H be a cyclic group generated by an element of order qM. Then $G \times H$ is a group satisfying the required condition.

These investigations were taken up on the suggestion of Prof. F. W. Levi, and I am grateful to him for his very helpful discussions.

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* Existence of such groups has been proved by F. W. Levi in the paper "Notes on group theory VII" which is likely to be published soon.

ERRATA

ON THE DIFFERENTIABILITY OF MONOTONE FUNCTIONS

By

P. D. Shukla

This Bulletin, Vol. 37, No. 1 (1945)

Page	*for*	*Read*
13	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} \neq 0$	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} > 0$

ON THE DIFFERENTIABILITY OF AN INDEFINITE INTEGRAL

By

P. D. Shukla

This Bulletin, Vol. 38, No. 1 (1946)

Page	*for*	*Read*
25 (foot-note)	(*i*) $F(x) = \int_0^x f(t)dt = O(x^\delta)$	$F(x) = \int_0^x f(t)dt = o(x^\delta)$
	(*ii*) $f(x) = 0$	$f(0) = 0$
27 and 30	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} \neq 0$	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} > 0$
30	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} = \frac{1}{3} \neq 0$	$\lim\limits_{r\to\infty} \frac{x_{r+1}}{x_r} = \frac{1}{3} > 0$

ON THE PETERSEN-MORLEY THEOREM—II

By

C. V. H. Rao

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1. A geometric proof of the theorem is recorded in the last volume (Rao, 1945). The available algebraic proofs are either too long or too deep. I now give a simple analytic proof based on a symbolism akin to that of Grassman, and proceed to show that the origin of the theorem is a certain proposition of incidences in S_5.

2. Start with a plane Π and a Desargues' configuration therein of two triangles ABC and $A'B'C'$ in perspective, with O for centre, and LMN for axis; with the lines $B'C$ and BC' meeting in P and so for Q and R.

Given such a configuration in the plane of x, y, z it is easy to assure oneself that the algebraic postulation can be so fixed as to satisfy the following three sets of requirements.

(α) $MQ+NR = 0$ passes through L,
(β) $OQ-OR = 0$ passes through L,
(γ_1) $OP+LP = B'CP$, and as a consequence of harmonics
(γ_2) $OP-LP = BC'P$.

Each of these statements carries with it two other similar statements. In other words, the left hand sides of the equations of the various lines mentioned can be taken with such multiplicative constants as to ensure the truth of these statements. This is a simple result of plane geometry to be verified in § 7.

3. Now let a, b, c be three arbitrary lines in space, drawn respectively through A, B, C; and a', b', c' the secants to pairs of them from A', B', C' respectively. Either triad fixes the other. Further let p be the common line of the planes $b'c$ and bc'; and so for q and r. Finally let E be the common point of the three planes $b'c$, $c'a$, $a'b$; and F the common point of the three planes bc', ca', ab'. Then the part of the figure in space generated by the lines a, b, c and (or) a', b', c' may equally be fixed by the prescription of the two points E and F; for the planes joining E to the triad of lines $B'C$, $C'A$, $A'B$ in the plane Π yield by their intersections with the planes joining F to the triad of lines BC', CA', AB' in the plane Π, precisely the two triads of lines a, b, c and a', b', c' as well as the triad p, q, r. For clearness $EC'A$ meets FAB' in a; the planes $EA'B$ and FCA' meet in a'; finally the planes $EB'C$ and FBC' meet in p.

4. Now consider the following table.

$I = Eb'cp$	$II = Ec'aq$	$III = Ea'br$
$I' = Fbc'p$	$II' = Fca'q$	$III' = Fab'r$

Let the symbols here such as I, I' stand for the left hand sides of the equations of the planes indicated, namely $Eb'cp$, $Fbc'p$ taken with such multiplicative constants

that on putting the variable t equal to zero we get the lines $B'CP$, $BC'P$ as fixed in (γ) of § 2. From the same reference, with the I and I' thus defined, it follows that $I = Op+Lp$ and $I' = Op-Lp$ where again Op and Lp are the left hand sides of the planes joining O, L respectively to the line p, taken in such wise that on putting the variable t equal to zero therein, we get the lines OP, LP as fixed in § 2.

5. Thus we have the scheme,

$I = Op+Lp$	$II = Oq+Mq$	$III = Or+Nr$
$I' = Op-Lp$	$II' = Oq-Mq$	$III' = Or-Nr$

Now it is seen from the Table that the planes II and III' have the line a in common, whence the plane La is a linear function of them; and is $II-III' = 0$ if we can show that $(Oq+Mq)_L$ equals $(Or-Nr)_L$: that is, since L lies in the plane t, we need to show that $(OQ+MQ)_L$ equals $(OR-NR)_L$. This is true in view of the conditions (α) and (β) in § 2 since $OQ-OR$ and $MQ+NR$ pass through L.

Again, the planes III and II' have the line a' common, whence the plane La' is a linear function of them and is $III-II' = 0$ if we can show that $(Or+Nr)_L$ equals $(Oq-Mq)_L$. This is so for the same reason as above.

Thus the plane La is $II-III' = 0$, and the plane La' is $III-II' = 0$.

6. Now let a'' be the secant from L to a, a'. It is then the common line of the planes La and La'. The plane joining O to a'' would then be that linear function of $II-III' = 0$ and $III-II' = 0$ which passes through O. The linear function sought would be $(II-III')-(III-II') = 0$ if we can show that $(II-III')_O$ equals $(III-II')_O$; in other words, the value of $Oq-Or+Mq+Nr$ and that of $Or-Oq+Mq+Nr$ at O are to be equal. This is certainly so for $Oq-Or$ passes through O. Thus the plane Oa'' is $(II+II')-(III+III') = 0$ which is $Oq-Or$; and obviously the three such planes add up to zero, showing they have a common line through O. Wherefore the three lines a'', b'', c'' have a common secant through O; which completes the proof of the theorem.

7. We pass to show that what is said in § 2 is possible, and begin with some preliminary remarks. The triangle ONR is self conjugate for every conic through A, A', B, B'; so also the triangle OMQ is self conjugate for every conic through A, A', C, C'; whence the lines MQ and NR intersect on AA' in the fourth harmonic of O. Moreover considering the triangle PQR and the secant LMN, the pole of this secant may be called U. Then the harmonic of U in P, QR is U_1 and is no other than the common point of MQ and NR. Thus O, A, A', U_1 all lie on one line, and the points A, A' are harmonically separated by the points O, U_1. Say $U \equiv U_1+U_2+U_3$, then P is U_2+U_3. Thus if we take A as $O+dU_1$—and hence by harmonics A' is $O-dU_1$—remembering that $QC'A$ and $RA'B$ are lines and that $Q = U_3+U_1$ and $R = U_1+U_2$, it follows that $C' = O-dU_3$ and $B = O+dU_2$; and by harmonics $C = O+dU_3$ and $B' = O-dU_2$.

Therefore, given PQR, LMN and O with the help of a single further constant d we define A, B, C as $O+dU_1$, $O+dU_2$, $O+dU_3$ and A', B', C' then arise by changing the sign of d.

Taking PQR as triangle of reference, LMN as unit line, and O as $a:b:c$, we may take

$$LP = -d(y+z),\quad MQ = -d(z+x),\quad NR = -d(x+y),$$

$$OP = cy-bz,\quad OQ = az-cx,\quad OR = bx-ay.$$

Then $MQ+NR = -d(2x+y+z)$ and $OQ-OR = -(b+c)x+a(y+z)$ both pass through L which is the common point of $x=0$ and $\sum x = 0$. Moreover $OP+LP = (c-d)y-(b+d)z$ which is $B'CP$ and hence by harmonics $OP-LP = (c+d)y-(b-d)z = BC'P$.

This verifies the assertion made in § 2.

8. In regard to the points E, F their joining line meets the plane PQR in a point whose harmonic wrt: E, F may be taken as S the fourth vertex of reference; so that we may take E as $x_0 : y_0 : z_0 : -t_0$ and F as $x_0 : y_0 : z_0 : t_0$. The equation of the plane $I = \sum b'c$ may then be put down as the plane of the three points E, B', C. We thus get the equations as under

$I = (c-d)(yt_0+y_0t)-(b+d)(zt_0+z_0t)$	$I' = (c+d)(yt_0-y_0t)-(b-d)(zt_0-z_0t)$
$II = (a-d)(zt_0+z_0t)-(c+d)(xt_0+x_0t)$	$II' = (a+d)(zt_0-z_0t)-(c-d)(xt_0-x_0t)$
$III = (b-d)(xt_0+x_0t)-(a+d)(yt_0+y_0t)$	$III' = (b+d)(xt_0-x_0t)-(a-d)(yt_0-y_0t)$

Defining H, K as the two points $a, b, c, \pm d$ the figure may be generated in terms of seven arbitrary points P, Q, R, S and E, H, U by means of harmonics and incidences; here U is the unit point in space.

The equations put down above help us to show that the final line, obtained as $Op = Oq = Or$, is the join of the point O with the point of co-ordinates y_0-z_0, z_0-x_0, x_0-y_0, $(\sum a/d)t_0$.

9. Since the proof depends ultimately on linear relations between six items like I, I' we may replace them by six variables like X, X' and take these as coordinates in a S_5.

Call the solids $Y=Z'=0$, $Z=X'=0$, $X=Y'=0$ the solids a, b, c respectively and take the solids a', b', c' as $Y'=Z=0$, $Z'=X=0$, $X'=Y=0$; let the plane $X-X'=Y-Z'=Z-Y'=0$ be called the plane L, and by symmetry call the plane $Y-Y'=Z-X'=X-Z'=0$ the plane M and let the plane $Z-Z'=X-Y'=Y-X'=0$ be called the plane N. Finally let the plane $X+X'=Y+Y'=Z+Z'=0$ be the O plane.

Then $Y-Z'=0$ is the prime through plane L and solid a; and $Z-Y'=0$ is the prime through plane L and solid a'. Through their common solid and the plane O passes the prime $(Y-Z')-(Z-Y')=0$ *i.e.*, $(Y+Y')-(Z+Z')=0$; and the three such primes have not only the plane O in common, but the solid $X+X'=Y+Y'=Z+Z'$; and this is equivalent to the fact that the plane O meets the three planes L, M, N in three collinear points. This is a simple result of incidences involving the six solids like a, a' and the four planes L, M, N, O and is the origin in S_6 of the theorem.

We pass to show that the section of the figure by an appropriate solid yields the theorem in S_3. Consider the quadric locus $\sum(X^2-2YZ)=\sum(X'^2-2Y'Z')$. This contains the planes L, M, N, O; passes through the unit point and there has for tangent prime $\sum X=\sum X'$, namely the prime containing the planes L, M, N. There are ∞^3 lines each of which is a secant to all three planes L, M, N and is thus a generator of the quadric. Through any one such generator $L_1M_1N_1$ there pass two generator planes, one of which meets the plane O in a point, say O_1. This plane Π can then be shown to be met by the six solids like a, a' in six points like A_1, A'_1 such that these six points together with L_1, M_1, N_1, O_1 make a Desargues' configuration in the plane Π.

Any arbitrary solid through the plane Π cuts out of the figure of six solids and four planes in S_6 the complete figure of the theorem in S_3, namely the six lines like a, a' besides the plane Desargues' figure. The choice of the generator line $L_1M_1N_1$ fixes the plane Desargues' figure and then the choice of the solid of section fixes the six lines like a, a'.

A remark has been made somewhere to the effect that every contribution of Morley to Geometry is a thing of beauty. That is certainly true about this theorem.

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ON COMPLETE PRIMITIVE RESIDUE SETS

By

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In a recent paper* S. Chowla and T. Vijayaraghavan proved that if $n = 2$ or has no primitive roots there exist suitable complete primitive residue sets † $r_1, r_2, \ldots, r_h$ and $s_1, s_2, \ldots, s_h$ such that $r_1s_1, r_2s_2, \ldots, r_hs_h$ too is a complete primitive residue set.

The purpose of this note is to discuss a type of such sets for $n = 2^m$, $m \geq 3$. Let

$$r_x \equiv 2k+2x-1 \quad (2^m); \qquad t_x \equiv 2k'+2x+1 \quad (2^m),^{**}$$
$$R_x \equiv 2k-(2x-1) \quad (2^m); \qquad T_x \equiv 2k'+2^{m-1}+2x+1 \quad (2^m),$$

where $1 \leq x \leq 2^{m-2}$ and $k \equiv k'$ (mod. 2).

It can be easily verified that (A) $r_1, r_2, \ldots, r_{h'}, R_1, R_2, \ldots, R_{h'}$ and (B) $t_1, t_2, \ldots, t_{h'}, T_1, T_2, \ldots, T_{h'}$, where $h' = 2^{m-2}$, are complete primitive residue sets.

Now construct

$$s_x \equiv \frac{t_x}{r_x} \text{ (mod. } 2^m),$$

$$S_x \equiv \frac{T_x}{R_x} \text{ (mod. } 2^m).$$

We prove that (A) and (C) $s_1, s_2, \ldots, s_{h'}, S_1, S_2, \ldots, S_{h'}$ are suitable sets satisfying the conditions of the theorem.

As (A) and $r_1s_1, r_2s_2, \ldots, r_{h'}s_{h'}, R_1S_1, R_2S_2, \ldots, R_{h'}S_{h'}$, *i.e.*, (A) and (B) are *c. p. r. s*'s it is only necessary to prove that (C) is a *c. p. r. s.*

In order to prove that it is sufficient to prove

I. All the s's and S's are prime to 2^m.

II. (i) no $s_{x_1} \equiv s_{x_2}$ (mod. n) if $x_1 \neq x_2$,

(ii) no $S_{x_1} \equiv S_{x_2}$ (mod. n) if $x_1 \neq x_2$,

(iii) no $s_{x_i} \equiv S_{x_j}$ (mod. n), x_i may or may not equal x_j.

I is trivial. The proofs of II (i), (ii) and (iii) are as follows.

Proof of II (i). Suppose there exists numbers x_1 and x_2, $1 \leq x_1, x_2 \leq 2^{m-2}$ and $x_1 \neq x_2$ such that

$$s_{x_1} \equiv s_{x_2} \text{ (mod. } 2^m)$$

then

$$\frac{t_{x_1}}{r_{x_1}} \equiv \frac{t_{x_2}}{r_{x_2}} \text{ (mod. } 2^m)$$

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† We mean by the *c. p. r. s.* what Hardy and Wright call complete set of residues prime to n.

** The possibility of the solution corresponding to $k = k' = 0$ was pointed out to me by Dr. S. Chowla.

or
$$\frac{2k'+2x_1+1}{2k+2x_1-1} \equiv \frac{2k'+2x_2+1}{2k+2x_2-1} \pmod{2^m}$$
i.e.,
$$4(x_1-x_2)(k-k'-1) \equiv 0 \pmod{2^m}$$
or
$$(x_1-x_2)(k-k'-1) \equiv 0 \pmod{2^{m-2}}.$$
But it is cleary impossible for $k-k'-1$ is odd and $0 < |x_1-x_2| \leq 2^{m-2}-1$. Therefore II (i) is true.

The proof of II(ii) is similar.

Proof of II (iii). Suppose there exist numbers x_i and x_j for which
$$s_{x_i} \equiv S_{x_j} \pmod{2^m}$$
then
$$\frac{2k'+2x_i+1}{2k+2x_i-1} \equiv \frac{2k'+2^{m-1}+1+2x_j}{2k-2x_j+1} \pmod{2^m}.$$
After a little simplification we have
$$4kx_i-4kx_j-4k'x_i-4k'x_j+4k'-8x_ix_j \equiv -2(2^{m-2}+1) \pmod{2^m},$$
which is clearly impossible because the $L.\ H.\ S$ and 2^m are multiples of 4 while $2(2^{m-2}-1)$ is not [because $m \geq 3$].

Therefore we see that (A) and (C) satisfy the conditions of the problem.

Changing R_x, S_x and T_x into $r_{2^{m-1}+1-x}$, $s_{2^{m-1}+1-x}$, and $t_{2^{m-1}+1-x}$ respectively we can show the three sets in a tabular form as follows.

$x =$	1	2	3	...	2^{m-2}	$2^{m-2}+1$	...	$2^{m-1}-2$	$2^{m-1}-1$	2^{m-1}
$r_x \equiv$	$2k+1$	$2k+3$	$2k+5$	...	$2k+2^{m-1}-1$	$2k-2^{m-1}+1$	...	$2k-5$	$2k-3$	$2k-1$
$s_x \equiv$	$\frac{t_1}{r_1}$	$\frac{t_2}{r_2}$	$\frac{t_3}{r_3}$	...	$\frac{t_{2^{m-2}}}{r_{2^{m-2}}}$	$\frac{t_{2^{m-2}+1}}{r_{2^{m-2}+1}}$	...	$\frac{t_{2^{m-1}-2}}{r_{2^{m-1}-2}}$	$\frac{t_{2^{m-1}-1}}{r_{2^{m-1}-1}}$	$\frac{t_{2^{m-1}}}{r_{2^{m-1}}}$
$t_x \equiv$	$2k'+3$	$2k'+5$	$2k'+7$	...	$2k'+2^{m-1}+1$	$2k'+2^{m-1}+2^{m-1}+1$ or $2k'+1$	...	$2k'+2^{m-1}+7$	$2k'+2^{m-1}+5$	$2k'+2^{m-1}+3$

Note:—It is interesting to observe that if we replace k' by $k'+2^{m-2}$, the series for t_x is reversed.

2. In the following discussion we take r_x, s_x and t_x less than 2^m.

Definition :—We shall call two systems of sets $r_1, r_2, \ldots, r_{2^{m-1}}$; $s_1, s_2, \ldots, s_{2^{m-1}}$; and $t_1, t_2, \ldots, t_{2^{m-1}}$ distinct if there exists at least one r, the two s's and hence the two t's corresponding to which in the two systems are different.

Theorem. *Corresponding to every m there are exactly 2^{2m-3} distinct systems of the type discussed in this note.*

LEMMA 1. Corresponding to each k there are exactly 2^{m-2} distinct systems.

When k is fixed there are exactly 2^{m-2} values of $k' \equiv k \pmod{2}$ such that no two of them are congruent to each other modulo 2^{m-1}.

For all these values of k', the t_1's are different for they are all congruent to numbers which are mutually incongruent modulo 2^m.

But if $k'' \equiv k' \pmod{2^{m-1}}$, all $t''_l = t'_l$ where t''_l and t'_l denote the lth. t's corresponding to k'' and k' respectively.

Therefore corresponding to each k there exist 2^{m-2} distinct systems.

LEMMA 2. If $k_1 \not\equiv k_2 \pmod{2^{m-1}}$ every system corresponding to $k = k_1$ is distinct from every system corresponding to $k = k_2$.

Proof. Let

$$r_1, r_2, \ldots, r_{2^{m-2}}, r_{2^{m-2}+1}, \ldots, r_{2^{m-1}}$$

$$s_1, s_2, \ldots, s_{2^{m-2}}, s_{2^{m-2}+1}, \ldots, s_{2^{m-1}}$$

and
$$t_1, t_2, \ldots, t_{2^{m-2}}, t_{2^{m-2}+1}, \ldots, t_{2^{m-1}}$$

be the system corresponding to k_1, k'_1, where k'_1 is any number $\equiv k_1 \pmod{2}$, and

$$R_1, R_2, \ldots, R_{2^{m-2}}, R_{2^{m-2}+1}, \ldots, R_{2^{m-1}}$$

$$S_1, S_2, \ldots, S_{2^{m-2}}, S_{2^{m-2}+1}, \ldots, S_{2^{m-1}}$$

and
$$T_1, T_2, \ldots, T_{2^{m-2}}, T_{2^{m-2}+1}, \ldots, T_{2^{m-1}}$$

be the system corresponding to k_2, k'_2, where k'_2 is any number $\equiv k_2 \pmod{2}$.

Let $r_{2^{m-2}} = R_b$, then $b \neq 2^{m-2}$. If not let $b = 2^{m-2}$, then $r_{2^{m-2}} = R_{2^{m-2}}$, therefore

$$2k_1 + 2^{m-1} - 1 \equiv 2k_2 + 2^{m-1} - 1 \pmod{2^m}$$

or
$$2(k_1 - k_2) \equiv 0 \pmod{2^m}$$

which is impossible because $k_1 \not\equiv k_2 \; (2^{m-1})$. Therefore $b \neq 2^{m-2}$.

Now there arise two cases:

$$(1) \quad b \neq 2^{m-1}, \text{ and } (2) \quad b = 2^{m-1}.$$

Case 1. $b \neq 2^{m-1}$, then $r_{2^{m-2}} = R_b$ and $r_{2^{m-2}+1} = R_{b+1}$, where $0 < b < b+1 \leq 2^{m-2}$ or $2^{m-2} < b < b+1 \leq 2^{m-1}$ (α).

If the two systems are not distinct

$$t_{2^{m-2}} = T_b \text{ and } t_{2^{m-2}+1} = T_{b+1}$$

or
$$t_{2^{m-2}} - t_{2^{m-2}+1} = T_b - T_{b+1}$$

i.e., $$2k_1'+1+2^{m-1}-2k_1'-1 \equiv \pm 2 \pmod{2^m}, \text{ [because of } (\alpha)]$$

or $$2^{m-1} \mp 2 \equiv 0 \ (2^m)$$

which is impossible because $m > 2$. Therefore in this case the two systems are distinct.

Case 2. If $b = 2^{m-1}$,

$$r_{2^{m-2}-1} = R_{2^{m-1}-1} \text{ and } r_{2^{m-2}} = R_{2^{m-1}}$$

If the two systems are not distinct

$$t_{2^{m-2}-1} = T_{2^{m-1}-1} \text{ and } t_{2^{m-2}} = T_{2^{m-1}}$$

or $$t_{2^{m-2}-1} - t_{2^{m-2}} = T_{2^{m-1}-1} - T_{2^{m-1}}$$

or $$2 \equiv -2 \pmod{2^m} \qquad \text{(see the Table)}$$

or $$4 \equiv 0 \pmod{2^m}$$

which is impossible because $m \geq 3$.

Therefore in all cases the two systems are distinct.

LEMMA 3. It can be easily verified that if $k_2 \equiv k_1 \pmod{2^{m-1}}$, the systems corresponding to (k_2, k') and (k_1, k') are not distinct.

Combining the above lemmas we can easily prove that the number of distinct systems of this type for every m is

$$2^{m-2} \cdot 2^{m-1} = 2^{2m-3}.$$

In the end I acknowledge with great pleasure my gratitude to Dr. S. Chowla, PH. D., whose encouragement made this note possible.

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NOTE ON THE STRESSES IN A SEMI-INFINITE PLATE PRODUCED BY A RIGID PUNCH ON THE STRAIGHT BOUNDARY

By

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Introduction

It appears that due attention has not been paid previously by investigators to problems of indentation produced by a rigid die pressing on the straight edge of a thin semi-infinite plate. The only case that has been treated so far is that of constant deflection produced by the die on a part of the straight edge of the plate. This particular problem was discussed by Sadowsky (1928) who used conformal mapping by which a circle was transformed to the loaded portion of the boundary. When the deflection of the loaded part is not constant, this method leads to considerable difficulty. A simple tentative method is employed in this note to deduce the results for various forms of indentation caused by a rigid punch on the straight edge.

1. Method of solution

Let the origin be taken on the straight edge along which the axis of y is taken. The axis of x is drawn into the plate perpendicular to this edge. The problem is considered to be one of 'generalized plane stress' so that if the normal load on the edge $x=0$ be given by

$$[\mathbf{Re}\, f(z)]_{x=0} \tag{1.1}$$

where **Re** denotes the real part and $z = x+iy$, then the average stresses $\breve{xx}$, $\breve{xy}$, $\breve{yy}$ are obtained as

$$\begin{aligned} \breve{xx} &= \mathbf{Re}[-xf'(z)+f(z)], \\ \breve{xy} &= \mathbf{Re}[-ixf'(z)], \\ \breve{yy} &= \mathbf{Re}[f(z)+xf'(z)]. \end{aligned} \tag{1.2}$$

These results were obtained previously by the author (Sen, 1938) in a simple manner. In this case we know that on neglecting the rigid body displacements, the average displacements u, v are given by

$$\begin{aligned} 2\mu u &= \mathbf{Re}[2(1-\bar{\sigma})F(z)-xf(z)], \\ 2\mu v &= \mathbf{Re}[-i(1-2\bar{\sigma})F(z)-ixf(z)], \end{aligned} \tag{1.3}$$

where μ is the modulus of rigidity, $\bar{\sigma}$ is a constant connected with Poisson's ratio σ by the relation $(1-\bar{\sigma}) = (1+\sigma)^{-1}$, and $F(z)$ is an analytic function of z such that

$$F(z) = \int f(z)dz. \tag{1.4}$$

Thus we find that on the edge $x = 0$, we can write

$$\begin{aligned} [\widehat{xx}]_{x=0} &= [\mathbf{Re}\, f(z)]_{x=0}, \\ [u]_{x=0} &= (2/E)[\mathbf{Re}\, F(z)]_{x=0} + C, \end{aligned} \tag{1.5}$$

where E is Young's modulus and C is a constant. It is to be noted that for a concentrated load P at the origin, we have

$$f(z) = -P/\pi z. \tag{1.6}$$

If the portion of the boundary pressed be given by $-b \leq y \leq b$, and the deformation is symmetrical about the origin, the problem reduces to that of finding $f(z)$ so that the normal load outside the pressed portion of the edge is zero, while the normal displacement u inside it, assumes the prescribed symmetrical form. In other words, we are to find $f(z)$ such that

$$[\mathbf{Re}\, f(z)]_{x=0} = 0 \quad \text{when } |y| > b \tag{1.7}$$

and the given symmetrical indentation is represented by

$$\frac{2}{E}\left[\mathbf{Re}\left\{\int f(z)dz\right\}\right]_{x=0} + C \quad \text{when} \quad |y| < b. \tag{1.8}$$

If P be the total load on the pressed part, $f(z)$ must tend to $-P/\pi z$ as z tends to infinity, confirming thereby that at a great distance, the effect is the same as that of a concentrated load at the origin.

2. Examples

(a) *Approximately circular indentation by a rigid punch*

Let

$$f(z) = -(2P/\pi b^2)[\sqrt{(b^2+z^2)} - z]. \tag{2.1}$$

Then

$$\begin{aligned} [\mathbf{Re}\, f(z)]_{x=0} &= -(2P/\pi b^2)\sqrt{(b^2-y^2)} \quad \text{when } -b \leq y \leq b, \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{when } y > b \text{ and } y < -b. \end{aligned} \tag{2.2}$$

This function satisfies the condition (1 7) and the total load on the pressed portion is of magnitude

$$\frac{2P}{\pi b^2}\int_{-b}^{b} \sqrt{(b^2-y^2)}dy = P.$$

Since in this case

$$F(z) = -\frac{P}{\pi b^2}[z\sqrt{(b^2+z^2)} + b^2 \sinh^{-1}(z/b) - z^2],$$

we obtain

$$[u]_{x=0} = u_0 = C - \frac{2P}{\pi b^2 E}y^2 \quad \text{when } -b < y < b.$$

From the condition that $u_0 = 0$ when $y = \pm b$, we have

$$u_0 = \frac{2P}{\pi b^2 E}(b^2 - y^2). \tag{2.3}$$

The curvature

$$\frac{1}{R} = -\frac{\partial^2 u_0}{\partial y^2} = \frac{4P}{\pi b^2 E} \quad \text{(a constant quantity).}$$

Again since

$$f(z) = -\frac{2P}{\pi b^2}\left[z\left(1+\frac{b^2}{2z^2}+\cdots\right)-z\right],$$

we find

$$f(z) \to -P/\pi z \quad \text{as} \quad z \to \infty.$$

(b) Constant deflection produced by a rigid punch

Let

$$f(z) = -\frac{P}{\pi\sqrt{(b^2+z^2)}}. \tag{2.4}$$

In this case

$$\begin{aligned}[\mathbf{Re}\, f(z)]_{x=0} &= -\frac{P}{\pi\sqrt{(b^2-y^2)}} && \text{when } -b<y<b \\ &= 0 && \text{when } y>b \text{ and } y<-b. \end{aligned} \tag{2.5}$$

Total pressure produced by the die is of magnitude

$$\frac{P}{\pi}\cdot\int_{-b}^{b}\frac{dy}{\sqrt{(b^2-y^2)}} = P.$$

$$\begin{aligned}[u]_{x=0} = u_0 &= -\frac{2P}{\pi E}\mathbf{Re}\left(\sinh^{-1}\frac{iy}{b}\right)+C \\ &= C, \quad \text{when } -b<y<b. \end{aligned} \tag{2.6}$$

At the edge of the die, that is, when $y = \pm b$, we find that the intensity of pressure as given by (2.5) becomes indefinitely large which shows that there is plastic flow at these points. From (2.4) it is evident that

$$f(z) \to -P/\pi z \quad \text{as } z \to \infty.$$

(c) A triangular indentation by a rigid punch

Let

$$f(z) = -\frac{P}{\pi b}\sinh^{-1}(b/z). \tag{2.7}$$

This gives us

$$\begin{aligned}[\mathbf{Re}\, f(z)]_{x=0} &= -\frac{P}{\pi b}\cosh^{-1}(b/y) && \text{when } |y|<b \\ &= 0 && \text{when } |y|>b. \end{aligned} \tag{2.8}$$

The magnitude of the total thrust on the pressed part is

$$\frac{P}{\pi b}\int_{-b}^{b}\cosh^{-1}(b/y)dy.$$

This is an improper integral of which the principal value is P.

Since in this case

$$F(z) = -\frac{P}{\pi b}\left[z \sinh^{-1}\frac{b}{z} + b \sinh^{-1}\frac{z}{b}\right],$$

we have even when the general value of the function is taken

$$[u]_{x=0} = u_0 = C - Ay, \quad (0 < y < b)$$
$$= C + Ay, \quad (-b < y < 0) \tag{2.9}$$

where A is a constant.

It is evident that the deflection is triangular in form. In this case also we find that

$$f(z) \to -\frac{P}{\pi z} \quad \text{as } z \to \infty.$$

When $y \to 0$, the intensity of pressure as given in (2.2) tends to an infinite value. Therefore, there must be yielding at the vertex of the triangle.

(d) Indentation in the form of a parabola of fourth degree

Suppose

$$f(z) = -\frac{4P}{3\pi b^4}[(b^2-2z^2)\sqrt{(b^2+z^2)}+2z^3]. \tag{2.10}$$

Since

$$f(z) = -\frac{4P}{3\pi b^4}\left[(b^2-2z^2)z\left(1+\frac{1}{2}\frac{b^2}{z^2}-\frac{1}{8}\frac{b^4}{z^4}+\cdots\right)+2z^3\right],$$

we find that

$$f(z) \to -\frac{P}{\pi z} \quad \text{as } z \to \infty.$$

We have in the present case

$$[\mathbf{Re}\, f(z)]_{x=0} = -\frac{4P}{3\pi b^4}[(b^2+2y^2)\sqrt{(b^2-y^2)}] \quad \text{when } |y| < b$$
$$= 0 \qquad \text{when } |y| > b. \tag{2.11}$$

The total load on the pressed area is of magnitude

$$\frac{4P}{3\pi b^4}\int_{-b}^{b} (b^2+2y^2)\sqrt{(b^2-y^2)}dy = P.$$

The deflection of the edge is given by

$$[u]_{x=0} = u_0 = C - \frac{4P}{3\pi b^4 E} y^4 \quad \text{when } |y| < b. \tag{2.12}$$

This result shows that the indentation is of the form of a parabola of fourth degree.

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MAHAVIRA'S DIOPHANTINE SYSTEM

By

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1. In his account of Mahavira's work in diophantine analysis, B. Datta[1] noted the system

$$m(x+y) = n(u+v), \qquad (1)$$
$$pxy = quv,$$

in which m, n, p, q are any constant (given) integers, and x, y, u, v are the unknowns. In the sense defined in a recent paper,[2] the system (1) is separable, and hence the complete solution in integers is readily obtainable. The complete integer solution falls into sets according to the divisors of m, n, p, q.

Let g be the G. C. D. of m, n, and h the G. C. D. of p, q. Then we may write $m = gm_1$, $n = gn_1$, $p = hp_1p_2$, $q = hq_1q_2$, all the letters denoting integers; $1 = (m_1, n_1) = (p_1p_2, q_1q_2)$. Each of the equations in (1) is of the multiplicative type. From the theory of such equations[2], the complete integer solution of the first is

$$x = n_1w, \quad u+v = m_1w, \qquad (2)$$

and that of the second is

$$x = q_1x_1x_2, \quad u = p_1x_1y_1, \qquad (3)$$
$$y = q_2y_1y_2, \quad v = p_2x_2y_2,$$

where w, x_1, x_2, y_1, y_2 are integer parameters. Since g, h, are determined when m, n, p, q are given, so are $m_1, n_1, p/h, q/h$, and p_1, p_2 and q_1, q_2 are any pairs of conjugate divisors of p/h and of q/h respectively.

Substituting from (3) into (2), we get the simultaneous system (4) for x_1, y_2, w,

$$p_1y_1.x_1 + p_2x_2.y_2 - m_1.w = 0, \qquad (4)$$
$$q_1x_2.x_1 + q_2y_1.y_2 - n_1.w = 0.$$

The solution of systems such as (4) is known[2]. There are two cases: the regular solution, in which not all second-order determinants in the coefficient matrix of (4) vanish; the singular solution, in which all three determinants vanish. Those solutions are respectively

$$x_1 = k(m_1q_2y_1 - n_1p_2x_2), \quad x_1 = p(q_2y_1w_3 - m_1w_2),$$
$$y_2 = k(n_1p_1y_1 - m_1q_1x_2), \quad y_2 = p(m_1w_1 - p_1y_1w_3),$$
$$w = k(p_1q_2y_1^2 - p_2q_1x_2^2), \quad w = p(q_2y_1w_1 - p_1y_1w_2),$$

in which x_2, y_1, w_1, w_2, w_3 are integer parameters; $m_1, n_1, p_1, p_2, q_1, q_2$ are as defined above, k is an arbitrary integer multiple of the reciprocal of the G. C. D. of the three ( )'s in the regular solution, for assigned values of the parameters x_2, y_1, and likewise

for p in the singular solution and assigned values of the parameters y_1, w_1, w_2, w_3. From these and (3), the complete integer solution of the system (1) is given by the regular solution

$$
\begin{aligned}
x &= kq_1x_2(m_1q_2y_1-n_1p_2x_2), \\
y &= kq_2y_1(n_1p_1y_1-m_1q_1x_2), \\
u &= kp_1y_1(m_1q_2y_1-n_1p_2x_2), \\
v &= kp_2x_2(n_1p_1y_1-m_1q_1x_2),
\end{aligned} \tag{5}
$$

and the singular solution

$$
\begin{aligned}
x &= pq_1x_2(q_2y_1w_3-m_1w_2), \\
y &= pq_2y_1(m_1w_1-p_1y_1w_3), \\
u &= pp_1y_1(q_2y_1w_3-m_1w_2), \\
v &= pp_2x_2(m_1w_1-p_1y_1w_3).
\end{aligned} \tag{6}
$$

As remarked, these solutions fall into sets according to the specified divisors of m, n, p, q. It is readily verified that (5), (6) satisfy (1). Not all solutions are obtained if k, p are merely arbitrary integers, and not as defined above.

2. Mahavira's system dates from the 9th century A. D. As noted by Datta, it escaped mention in Dickson's *History*[3] in the place where it would naturally be looked for, and where the special cases named after Heron (1st century B. C.) and Planude (c. 1260-1310) are discussed. P. Tannery (1882) considered the special case $n = 1$, $p = 1$ of Mahavira's (1), and stated, without proof and with no indication of the means by which they were obtained, formulas (containing a misprint) purporting to give the complete integer solution of this special case, which includes Heron's and Planude's problems. Tannery's formulas are inexplicit and somewhat awkward to verify; they are equivalent to simple explicit formulas obtained by the general theory of separable systems. In contrast to the solution of (1), Tannery's special case requires no such step as (4), and the final formulas do not involve G. C. D.'s as for k, p in (5), (6).

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ON SEMIGROUPS II

By

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This paper deals with the homomorphisms which map systems in which a composition is defined on semigroups and with homomorphisms mapping semigroups on groups. There exist always such mapings, *e.g.*, the "trivial" mapping of the whole system on a single element. In some cases, the trivial mapping is the only homomorphism which maps the given system (semigroup) on a semigroup (group). It will be shown that all the semigroups which are homomorphic to a system W are also homomorphic to a semigroup $S(W)$ which is uniquely determined in the sense of isomorphism. The corresponding statement for the homomorphisms of semigroups on groups is in general not correct. For the determination of these homomorphisms, the notion of normal sub-semigroup, which was introduced in an earlier paper (Levi, 1944) is important.

§ 1. Let W be a set in which a non-associative multiplication is defined. W is mapped by homomorphisms $H_1, H_2, \ldots$ on semigroups $S_1, S_2, \ldots$. Each H determines a resolution of W into disjoint classes such that every class is mapped on the same element of the corresponding semigroup. S_1 is homomorphic to S_2 if and only if the resolution generated by H_2 is a refinement of the resolution generated by H_1. Two elements a and a' of W will be considered as congruent if and only if for every H, the elements a and a' belong to the same class. Denote the congruence classes by A, B, C and let $a, a' \in A$; $b, b' \in B$; $c \in C$. Then for every H, ab and $a'b'$ belong to the same class and similarly $(ab)c$ and $a(bc)$. Hence the congruence classes form a semigroup $S(W)$ which is homomorphic to W and is therefore generated by some H. The resolution into $A, B, C, \ldots$ is a refinement of every resolution generated by an H and therefore $S(W)$ is homomorphic to every $S_1, S_2, \ldots$. Hence:

Theorem 1. *When W is a set in which a non associative multiplication is defined, there exists a semigroup $S(W)$ such that every semigroup which is homomorphic to W is also homomorphic to $S(W)$.*

Obviously $S(W)$ is uniquely determined in the sense of isomorphism. When $S(W)$ is of order 1, only the trivial homomorphism H exists.

§ 2. For the homomorphic mapping of semigroups on groups the method used in §1 is not applicable. Consider, *e.g.*, the additive semigroup formed by the positive integral numbers; it can be mapped on the cyclic groups of finite order only and among these, there is no group which is homomorphic to all the other ones.

Let the semigroup S be mapped by a homomorphism H on a group G and let N be the subset of S which is mapped on the unitelement $\mathbf{1}$ of G. If b, $c \in N$, then H maps bc on $\mathbf{1}$ and therefore $bc \in N$; hence N is a sub-semigroup of S. If $a \in S$ is mapped by H on

$\alpha \in G$, then there exists an $a' \in S$ which is mapped on α^{-1} and therefore $aa' \in N$. Thus every $a \in S$ is a left hand factor of some element of N. We shall call a subgroup with that property a *complete* sub-semigroup of S. On the other hand when $aa' \in N$, it follows that H maps a and a' on elements which are inverse one to another. Hence $aa' \in N$ implies $a'a \in N$. Suppose now that ac, $b \in N$, then $ca \in N$ and therefore $abca$ is mapped by H on the same element of G as a; hence $abc \in N$. In this way one shows that when any two of the elements abc, ac, b belong to N, all three of them belong to N. Sub-semigroups N with the properties have been called (Levi, 1944, p. 141-42) *normal* sub-semigroups. Thus the original of 1 in S is a normal and complete sub-semigroup of S. Now we suppose that N is an arbitrary normal and complete sub-semigroup of S and show that there exists a homomorphism of S on a group for which N is the original of the unit-element. For this purpose, we introduce an equivalence relation between the elements of S by defining $a \backsim b$ if there exists an element a' such that aa' and ba' belong to N. This definition is reflexive and symmetric. To prove the transitivity, suppose aa', ba', bb', $cb' \in N$, then $a'b$, $aa'bb' \in N$ and therefore $ab' \in N$. Hence $a \backsim b$, $b \backsim c$ implies $a \backsim c$. Suppose now that $x \backsim y$, say xx', $yx' \in N$, then $axx'a'$, $byx'a' \in N$; hence $ax \backsim by$. The classes of equivalent elements form therefore a semigroup G which is homomorphic to S. However N is the unitelement of G and the class represented by a' is inverse to the class represented by a. Hence G is a group. On the other hand when S is mapped homomorphically on a group such that N is mapped on the unitelement, then a and a' must be mapped on inverse groupelements and therefore equivalent elements of S represent the same groupelement. If a and d are non-equivalent but are mapped on the same groupelement, then da' does not belong to N but is mapped on the unit-element of the group. The homomorphism mapping S on a group is therefore uniquely determined by N. Hence:

Theorem 2 *The homomorphisms H which map a semigroup S on a group, are uniquely determined by the complete and normal sub-semigroups N of S in such a way that N consists of those and only those elements of S which are mapped by H on the unitelement of the group.*

This theorem is a generalisation of Theorem 3 of the first note. N must contain all the idempotent elements of S, in particular all the left or right zeros. Therefore if S contains any zero, then $N = S$ and there exists no non-trivial mapping of S on a group. If N' is normal but non-complete, then the factors of the elements of N' form a proper sub-semigroup S' of S and S' can be mapped homomorphically on a group such that N' is mapped on the unitelement. If W is mapped by a homomorphism on a group G, then G is homomorphic to $S(W)$ and the mapping is therefore determined by the complete and normal sub-semigroups of $S(W)$.

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ON A TYPE OF SERIES INVOLVING THE PARTITION FUNCTION WITH APPLICATIONS TO CERTAIN CONGRUENCE RELATIONS

By

D. B. Lahiri

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1. In this paper I shall show that the sum $s_r(n)$ of the series

$$1^r.p(n-1)+2^r.p(n-2)-5^r.p(n-5)-7^r.p(n-7)+12^r.p(n-12)+15^r.p(n-15)-\cdots,$$

where 1, 2, 5, 7, 12, 15, . . . are the numbers $\frac{1}{2}n(3n\pm 1)$, and where $p(n)$ is the number of unrestricted partitions of n, $p(0)$ being supposed to be unity, can be expressed as a simple "divisor function" of n, when $r \leq 5$. I shall in fact prove

Theorem I. $s_1(n) = \sigma_1(n),$ (1.1)

$$12s_2(n) = -5\sigma_3(n)+(18n-1)\sigma_1(n), \tag{1.2}$$

$$192s_3(n) = 7\sigma_5(n)-10(15n-1)\sigma_3(n)+(360n^2-36n+1)\sigma_1(n), \tag{1.3}$$

$$3456s_4(n) = -5\sigma_7(n)+21(14n-1)\sigma_5(n)-15(252n^2-30n+1)\sigma_3(n) + (7560n^3-1080n^2+54n-1)\sigma_1(n), \tag{1.4}$$

$$331776s_5(n) = 11\sigma_9(n)-50(27n-2)\sigma_7(n)+210(216n^2-28n+1)\sigma_5(n) -100(4536n^3-756n^2+45n-1)\sigma_3(n)+5(163296n^4 -30240n^3+2160n^2-72n+1)\sigma_1(n), \tag{1.5}$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. It may be added here that when $r > 5$ our method does not enable us to determine the *sum* of the series in terms of $\sigma_k(n)$, where of course by the word *sum* is meant an expression in which the number of terms is independent of n.

From the above theorem I shall deduce a number of recurrence congruence relations analogous to those obtained by Ramanujan (1921) of the type

$$p(n-2)-p(n-51)-p(n-100)+p(n-247)+\cdots \equiv n^2\sigma_1(n)-n\sigma_3(n) \pmod 7,$$

the numbers 2, 51, 100, 247, . . . being those of the forms

$$\tfrac{1}{2}(7n-1)(21n-4),\quad \tfrac{1}{2}(7n+1)(21n+4).$$

Somewhat similar congruences have been considered by Simons (1944).

I have also indicated how each of these congruences corresponding to moduli 5, 7, 11 may be used to derive Ramanujan's famous congruences

$$p(5m+4) \equiv 0 \pmod 5,$$

$$p(7m+5) \equiv 0 \pmod 7,$$

$$p(11m+6) \equiv 0 \pmod{11}.$$

SUMMATION OF THE SERIES

2. Our method of summation depends upon a simple adaptation of certain results due to Ramanujan (1916) which I quote in some detail for ready reference. He writes

$$P = 1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\cdots\right),$$

$$Q = 1+240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\cdots\right),$$

$$R = 1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\cdots\right),$$

and denotes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn} = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n, \quad s > r,$$

by $\Phi_{r,s}(x)$; we shall however drop the x and write simply $\Phi_{r,s}$. Ramanujan then establishes the following:

TABLE I

$$\begin{aligned} 1-\ 24\Phi_{0,1} &= P,\\ 1+240\Phi_{0,3} &= Q,\\ 1-504\Phi_{0,5} &= R,\\ 1+480\Phi_{0,7} &= Q^2,\\ 1-264\Phi_{0,9} &= QR. \end{aligned}$$

TABLE II

$$\begin{aligned} 288\Phi_{1,2} &= Q-P^2,\\ 720\Phi_{1,4} &= PQ-R,\\ 1008\Phi_{1,6} &= Q^2-PR,\\ 720\Phi_{1,8} &= PQ^2-QR. \end{aligned}$$

TABLE III

$$\begin{aligned} 1728\Phi_{2,3} &= 3PQ-2R-P^3,\\ 1728\Phi_{2,5} &= P^2Q-2PR+Q^2,\\ 1728\Phi_{2,7} &= 2PQ^2-P^2R-QR,\\ 6912\Phi_{3,4} &= 6P^2Q-8PR+3Q^2-P^4,\\ 3456\Phi_{3,6} &= P^3Q-3P^2R+3PQ^2-QR,\\ 20736\Phi_{4,5} &= 15PQ^2-20P^2R+10P^3Q-4QR-P^5. \end{aligned}$$

3. We shall make use of the above tables to prove the LEMMA:

$$\begin{aligned} \Phi_{0,1}^2 &= \tfrac{1}{12}\Phi_{0,1}-\tfrac{1}{2}\Phi_{1,2}+\tfrac{5}{12}\Phi_{0,3},\\ \Phi_{0,1}.\Phi_{1,2} &= \tfrac{1}{24}\Phi_{1,2}-\tfrac{1}{4}\Phi_{2,3}+\tfrac{5}{24}\Phi_{1,4},\\ \Phi_{0,1}.\Phi_{2,3} &= \tfrac{1}{24}\Phi_{2,3}-\tfrac{1}{6}\Phi_{3,4}+\tfrac{1}{8}\Phi_{2,5},\\ \Phi_{0,1}.\Phi_{3,4} &= \tfrac{1}{24}\Phi_{3,4}-\tfrac{1}{8}\Phi_{4,5}+\tfrac{1}{12}\Phi_{3,6},\\ \Phi_{0,1}.\Phi_{0,3} &= -\tfrac{1}{240}\Phi_{0,1}+\tfrac{1}{24}\Phi_{0,3}-\tfrac{1}{8}\Phi_{1,4}+\tfrac{7}{80}\Phi_{0,5},\\ \Phi_{0,1}.\Phi_{1,4} &= \tfrac{1}{24}\Phi_{1,4}-\tfrac{1}{10}\Phi_{2,5}+\tfrac{7}{120}\Phi_{1,6},\\ \Phi_{0,1}.\Phi_{2,5} &= \tfrac{1}{24}\Phi_{2,5}-\tfrac{1}{12}\Phi_{3,6}+\tfrac{1}{24}\Phi_{2,7},\\ \Phi_{0,1}.\Phi_{0,5} &= \tfrac{1}{504}\Phi_{0,1}+\tfrac{1}{24}\Phi_{0,5}-\tfrac{1}{12}\Phi_{1,6}+\tfrac{5}{126}\Phi_{0,7},\\ \Phi_{0,1}.\Phi_{1,6} &= \tfrac{1}{24}\Phi_{1,6}-\tfrac{1}{14}\Phi_{2,7}+\tfrac{5}{168}\Phi_{1,8},\\ \Phi_{0,1}.\Phi_{0,7} &= -\tfrac{1}{480}\Phi_{0,1}+\tfrac{1}{24}\Phi_{0,7}-\tfrac{1}{16}\Phi_{1,8}+\tfrac{11}{480}\Phi_{0,9}. \end{aligned}$$

Ramanujan's tables give us the following and the proof is fairly obvious.

$$\Phi_{0,1}^2 = \frac{1}{24^2}(1-P)^2 = \frac{1}{24^2}[2(1-P)-(Q-P^2)+(Q-1)];$$

$$\Phi_{0,1}.\Phi_{1,2} = \frac{1}{24}(1-P)\cdot\frac{1}{288}(Q-P^2) = \frac{1}{24.288}[(Q-P^2)-(3PQ-2R-P^3)+2(PQ-R)];$$

$$\Phi_{0,1}\,\Phi_{2,3} = \frac{1}{24.1728}[(3PQ-2R-P^3)-(6P^2Q-8PR+3Q^2-P^4)+3(P^2Q-2PR+Q^2)];$$

$$\Phi_{0,1}.\Phi_{3,4} = \frac{1}{24.6912}[(6P^2Q-8PR+3Q^2-P^4)-(15PQ^2-20P^2R+10P^3Q-4QR-P^5) +4(P^3Q-3P^2R+3PQ^2-QR)];$$

$$\Phi_{0,1}.\Phi_{0,3} = \frac{1}{24.240}[-(1-P)+(Q-1)-(PQ-R)+(1-R)];$$

$$\Phi_{0,1}.\Phi_{1,4} = \frac{1}{24.720}[(PQ-R)-(P^2Q-2PR+Q^2)+(Q^2-PR)];$$

$$\Phi_{0,1}.\Phi_{2,5} = \frac{1}{24.1728}[(P^2Q-2PR+Q^2)-(P^3Q-3P^2R+3PQ^2-QR)+(2PQ^2-P^2R-QR)];$$

$$\Phi_{0,1}.\Phi_{0,5} = \frac{1}{24.504}[(1-P)+(1-R)-(Q^2-PR)+(Q^2-1)];$$

$$\Phi_{0,1}.\Phi_{1,6} = \frac{1}{24.1008}[(Q^2-PR)-(2PQ^2-P^2R-QR)+(PQ^2-QR)];$$

$$\Phi_{0,1}.\Phi_{0,7} = \frac{1}{24.480}[-(1-P)+(Q^2-1)-(PQ^2-QR)-(1-QR)].$$

4. I shall now explain certain notations which will be employed. The coefficients a_n are defined by

$$(1-x)(1-x^2)(1-x^3)\cdots = 1+\sum_0^\infty(-1)^n\{x^{\frac{1}{2}n(3n-1)}+x^{\frac{1}{2}n(3n+1)}\} = \sum_0^\infty a_n x^n.$$

The product

$$\sum_0^\infty n^r a_n x^n.\ \sum_0^\infty n^s p(n)x^n,$$

will be denoted by $u_{r,s}$, n^r being supposed to be unity when $n=r=0$. We have now, the following well-known result,

$$u_{0,0} = 1.$$

Also

$$u_{0,1} = \sum_0^\infty a_n x^n \sum_0^\infty np(n)x^n = \frac{\sum_0^\infty np(n)x^n}{\sum_0^\infty p(n)x^n} = x\frac{d}{dx}\left[\log\left\{\sum_0^\infty p(n)x^n\right\}\right]$$

$$= x\frac{d}{dx}[\log\{(1-x)(1-x^2)(1-x^3)\cdots\}^{-1}] = \sum_1^\infty\sigma_1(n)x^n = \Phi_{0,1}.$$

Also it can be easily seen that

$$\left(x\frac{d}{dx}\right).\ u_{r,0} = u_{r+1,0}+u_{r,1} = u_{r+1,0}+u_{r,0}.\ u_{0,1}.$$

Hence

$$u_{r+1,0} = \left[x\frac{d}{dx} - \Phi_{0,1}\right].u_{r,0}.$$

Thus using the operational symbol $x\dfrac{d}{dx} - \Phi_{0,1}$

$$u_{r,0} = \left[x\frac{d}{dx} - \Phi_{0,1}\right]^r . 1.$$

Giving r the values 1, 2, 3, 4, 5 successively, using our Lemma and remembering that

$$\left(x\frac{d}{dx}\right). \Phi_{r,s} = \Phi_{r+1, s+1},$$

we get

$$\begin{aligned}
u_{1,0} &= -\Phi_{0,1};\\
u_{2,0} &= -\Phi_{1,2} + \Phi_{0,1}^2 = \tfrac{1}{12}\Phi_{0,1} - \tfrac{3}{2}\Phi_{1,2} + \tfrac{5}{12}\Phi_{0,3};\\
u_{3,0} &= \tfrac{1}{12}\Phi_{1,2} - \tfrac{3}{2}\Phi_{2,3} + \tfrac{5}{12}\Phi_{1,4} - \Phi_{0,1}\{\tfrac{1}{12}\Phi_{0,1} - \tfrac{3}{2}\Phi_{1,2} + \tfrac{5}{12}\Phi_{0,3}\},\\
&= -\tfrac{1}{192}\Phi_{0,1} + \tfrac{3}{16}\Phi_{1,2} - \tfrac{15}{8}\Phi_{2,3} - \tfrac{5}{96}\Phi_{0,3} + \tfrac{25}{32}\Phi_{1,4} - \tfrac{7}{192}\Phi_{0,5};\\
u_{4,0} &= \tfrac{1}{3456}\Phi_{0,1} - \tfrac{1}{64}\Phi_{1,2} + \tfrac{5}{16}\Phi_{2,3} - \tfrac{35}{16}\Phi_{3,4} + \tfrac{5}{1152}\Phi_{0,3} - \tfrac{25}{192}\Phi_{1,4}\\
&\qquad + \tfrac{35}{32}\Phi_{2,5} + \tfrac{7}{1152}\Phi_{0,5} - \tfrac{49}{576}\Phi_{1,6} + \tfrac{5}{3456}\Phi_{0,7};\\
u_{5,0} &= -\tfrac{5}{331776}\Phi_{0,1} + \tfrac{5}{4608}\Phi_{1,2} - \tfrac{25}{768}\Phi_{2,3} + \tfrac{175}{384}\Phi_{3,4} - \tfrac{315}{128}\Phi_{4,5}\\
&\qquad - \tfrac{95}{82944}\Phi_{0,3} + \tfrac{125}{9216}\Phi_{1,4} - \tfrac{175}{768}\Phi_{2,5} + \tfrac{175}{128}\Phi_{3,6} - \tfrac{35}{55296}\Phi_{0,5} + \tfrac{245}{13824}\Phi_{1,6}\\
&\qquad - \tfrac{35}{256}\Phi_{2,7} - \tfrac{23}{82944}\Phi_{0,7} + \tfrac{25}{6144}\Phi_{1,8} - \tfrac{11}{331776}\Phi_{0,9}.
\end{aligned}$$

It is now quite simple to prove Theorem I; we have just to equate the coefficients of x^n on both sides of the identities established above.

Recursive congruence properties of $p(n)$

5. I shall now deduce from Theorem I a number of recursive congruence properties of the Ramanujan type, an example of which has already been quoted at the beginning of this paper. It will be convenient to employ the following notation. $I(n)$ will be used to denote the familiar expression

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots$$

which is the coefficient of x^n in the product $u_{0,0}$, and is vanishing if $n > 0$, as $u_{0,0} = 1$. Further $I_{am+b}(n)$ will denote the series obtained by picking out those terms $\pm p(n-r)$ of $I(n)$ in which r is of the form $am+b$. I shall now establish the following:

Theorem II.

$$\begin{aligned}
I_{2m}(n) &= p(n) - p(n-2) - p(n-12) + \cdots,\\
&\equiv -\tfrac{1}{3456}\{5\sigma_7(n) - 21(14n-1)\sigma_5(n) + 15(252n^2 - 30n + 1)\sigma_3(n)\\
&\qquad - (7560n^3 - 1080n^2 + 54n - 1)\sigma_1(n)\} \pmod{16}, && (2.01)\\
&\equiv -\tfrac{1}{12}\{5\sigma_3(n) - (18n-1)\sigma_1(n)\} \pmod{4}, && (2.02)\\
&\equiv \sigma_1(n) \pmod{2}; && (2.03)
\end{aligned}$$

$$
\begin{aligned}
I_{2m+1}(n) &= -p(n-1)+p(n-5)+p(n-7)-\cdots, \\
&\equiv \tfrac{1}{3456}\{5\sigma_7(n)-21(14n-1)\sigma_5(n)+15(252n^2-30n+1)\sigma_3(n) \\
&\qquad -(7560n^3-1080n^2+54n-1)\sigma_1(n)\} \pmod{16}, \qquad (2.11)\\
&\equiv \tfrac{1}{12}\{5\sigma_3(n)-(18n-1)\sigma_1(n)\} \pmod 4, \qquad (2.12)\\
&\equiv -\sigma_1(n) \pmod 2. \qquad (2.13)
\end{aligned}
$$

Since $(2m)^4 \equiv 0$; and $(2m+1)^4 \equiv 1 \pmod{16}$ we get

$$s_4(n) \equiv -I_{2m+1}(n) \pmod{16},$$

and by Theorem I (1.4) we get Theorem II (2.11). For weaker congruences to moduli 4 and 2 the right hand side can be replaced by simpler expressions $\frac{1}{12}\{5\sigma_3(n)-(18n-1)\sigma_1(n)\}$ and $-\sigma_1(n)$ respectively, as can be directly seen if we employ (1.2) and (1.1). (2.0) follows from (2.1) as

$$I_{2m}(n) = I(n)-I_{2m+1}(n) = -I_{2m+1}(n).$$

Theorem III.

$$I_{3m}(n) = p(n)-p(n-12)-p(n-15)+\cdots \equiv \tfrac{1}{3}\{4\sigma_3(n)-\sigma_1(n)\} \pmod 3; \qquad (3.0)$$

$$I_{3m+1}(n) = -p(n-1)+p(n-7)+p(n-22)-\cdots \equiv \tfrac{1}{3}\{\sigma_3(n)-4\sigma_1(n)\} \pmod 3; \qquad (3.1)$$

$$I_{3m+2}(n) = -p(n-2)+p(n-5)+p(n-26)-\cdots \equiv \tfrac{4}{3}\{\sigma_3(n)-\sigma_1(n)\} \pmod 3. \qquad (3.2)$$

It is easily seen that

$$-s_1(n) \equiv I_{3m+1}(n)+2I_{3m+2}(n) \pmod 3,$$

and

$$-s_2(n) \equiv I_{3m+1}(n)+I_{3m+2}(n) \pmod 3.$$

Thus

$$
\begin{aligned}
I_{3m+2}(n) &\equiv s_2(n)-s_1(n) \pmod 3,\\
&\equiv \tfrac{1}{12}\{-5\sigma_3(n)+(18n-13)\sigma_1(n)\} \equiv \tfrac{4}{3}\{\sigma_3(n)-\sigma_1(n)\}.
\end{aligned}
$$

Hence

$$I_{3m+1}(n) \equiv \tfrac{1}{3}\{\sigma_3(n)-4\sigma_1(n)\} \pmod 3;$$

and finally

$$I_{3m}(n) = -I_{3m+2}(n)-I_{3m+1}(n) \equiv \tfrac{1}{3}\{4\sigma_3(n)-\sigma_1(n)\} \pmod 3.$$

Theorem IV.

$$
\begin{aligned}
I_{4m}(n) &= p(n)-p(n-12)-p(n-40)+\cdots,\\
&\equiv \tfrac{1}{384}\{7\sigma_5(n)-10(15n-17)\sigma_3(n)+3(120n^2-204n-53)\sigma_1(n)\} \pmod 2; \qquad (4.0)\\
I_{4m+1}(n) &= -p(n-1)+p(n-5)+p(n-57)-\cdots,\\
&\equiv \tfrac{1}{384}\{7\sigma_5(n)-10(15n+7)\sigma_3(n)+3(120n^2+84n-5)\sigma_1(n)\} \pmod 2; \qquad (4.1)\\
I_{4m+2}(n) &= -p(n-2)+p(n-22)+p(n-26)-\cdots,\\
&\equiv \tfrac{1}{384}\{7\sigma_5(n)-10(15n-1)\sigma_3(n)+(360n^2-36n-191)\sigma_1(n)\} \pmod 2; \qquad (4.2)\\
I_{4m+3}(n) &= p(n-7)-p(n-15)-p(n-35)+\cdots,\\
&\equiv \tfrac{1}{384}\{7\sigma_5(n)-30(5n-3)\sigma_3(n)+(360n^2-324n+17)\sigma_1(n)\} \pmod 2. \qquad (4.3)
\end{aligned}
$$

This theorem evidently is a consequence of (2.12) and

$$I_{4m+1}+2I_{4m+2}-I_{4m+3} \equiv -s_1(n) \pmod 4,$$

$$I_{4m+1}-I_{4m+3} \equiv -s_3(n) \pmod 4.$$

Theorem V.

$$I_{5m}(n) = p(n)+p(n-5)-p(n-15)-\dots \equiv 3(n+1)\sigma_1(n) \pmod 5; \tag{5.0}$$
$$I_{5m+1}(n) = -p(n-1)+p(n-26)+p(n-51)-\dots \equiv -n\sigma_1(n) \pmod 5; \tag{5.1}$$
$$I_{5m+2}(n) = -p(n-2)+p(n-7)-p(n-12)+\dots \equiv 3(n-1)\sigma_1(n) \pmod 5; \tag{5.2}$$
$$I_{5m+3}(n) = 0; \tag{5.3}$$
$$I_{5m+4}(n) = 0. \tag{5.4}$$

It is interesting to note that here $I_{5m+3}(n)$ and $I_{5m+4}(n)$ are both vanishing Now it is easily seen that

$$-s_1(n) \equiv I_{5m+1}(n)+2I_{5m+2}(n), \pmod 5$$

and

$$-s_2(n) \equiv I_{5m+1}(n)-I_{5m+2}(n), \pmod 5$$

Hence

$$3I_{5m+2}(n) \equiv s_2(n)-s_1(n),$$

or,

$$I_{5m+2}(n) \equiv 2\{s_2(n)-s_1(n)\} \equiv 2[\tfrac{1}{12}\{-5\sigma_3(n)+(18n-1)\sigma_1(n)\}-\sigma_1(n)],$$
$$\equiv \tfrac{1}{6}\{-5\sigma_3(n)+(18n-13)\sigma_1(n)\},$$

or,

$$I_{5m+2}(n) \equiv 3(n-1)\sigma_1(n) \pmod 5.$$

Thus

$$I_{5m+1}(n) \equiv -s_1(n)-2I_{5m+2}(n) \pmod 5,$$
$$\equiv -n\sigma_1(n).$$

Finally

$$I_{5m}(n) = -I_{5m+1}(n)-I_{5m+2}(n) \equiv 3(n+1)\sigma_1(n) \pmod 5.$$

Congruence (5.1) was previously obtained by Ramanujan.

Theorem VI.

$$I_{7m}(n) = p(n)+p(n-7)-p(n-35)-\dots,$$
$$\equiv 2(n-2)\sigma_3(n)-2(n^2-3n-3)\sigma_1(n) \pmod 7; \tag{7.0}$$
$$I_{7m+1}(n) = -p(n-1)-p(n-15)+p(n-22)+\dots,$$
$$\equiv 2(n-1)\sigma_3(n)-2(n+1)^2\sigma_1(n) \pmod 7; \tag{7.1}$$
$$I_{7m+2}(n) \equiv -p(n-2)+p(n-51)+p(n-100)-\dots,$$
$$\equiv n\sigma_3(n)-n^2\sigma_1(n) \pmod 7; \tag{7.2}$$
$$I_{7m+3}(n) = 0; \tag{7.3}$$
$$I_{7m+4}(n) = 0; \tag{7.4}$$
$$I_{7m+5}(n) = p(n-5)-p(n-12)+p(n-26)-\dots,$$
$$\equiv 2(n+3)\sigma_3(n)-2(n^2+n+2)\sigma_1(n) \pmod 7; \tag{7.5}$$
$$I_{7m+6}(n) \equiv 0. \tag{7.6}$$

Here again we have the following with respect to modulus 7

$$-s_1(n) \equiv I_{7m+1}(n)+2I_{7m+2}(n)-2I_{7m+5}(n),$$
$$-s_2(n) \equiv I_{7m+1}(n)-3I_{7m+2}(n)-3I_{7m+5}(n),$$
$$-s_3(n) \equiv I_{7m+1}(n)+I_{7m+2}(n)-I_{7m+5}(n),$$

which imply

$$I_{7m+1}(n) \equiv -2s_3(n)+s_1(n),$$
$$I_{7m+2}(n) \equiv -s_3(n)-s_2(n)+2s_1(n).$$
$$I_{7m+5}(n) \equiv -2s_3(n)-s_2(n)+3s_1(n).$$

Thus

$$I_{7m+1}(n) \equiv -\tfrac{2}{192}\{7\sigma_5(n)-10(15n-1)\sigma_3(n)+(360n^2-36n+1)\sigma_1(n)\}+\sigma_1(n),$$

which easily reduces to

$$I_{7m+1}(n) \equiv 2(n-1)\sigma_3(n)-2(n+1)^2\sigma_1(n).$$

Results (7.2)—(7.6) similarly follow, and result (7.0), of course, as in similar situations for other moduli, is deducible from (7.1)—(7.6). Congruence (7.2) was proved by Ramanujan.

Theorem VII.

$$I_{11m}(n) = p(n)+p(n-22)-p(n-77)- \cdots,$$
$$\equiv (n-3)\sigma_7(n)-(5n^2+4n+2)\sigma_5(n)-(5n^3+2n^2-2n-4)\sigma_3(n)$$
$$-(2n^4-4n^3+3n^2-n-4)\sigma_1(n); \quad (11.0)$$

$$I_{11m+1}(n) = -p(n-1)-p(n-12)+p(n-100)+ \cdots,$$
$$\equiv (n+2)\sigma_7(n)-(5n^2+n+3)\sigma_5(n)-(5n^3-5n^2-3n+2)\sigma_3(n)$$
$$-(2n^4-n^3+5n^2-5n-5)\sigma_1(n); \quad (11.1)$$

$$I_{11m+2}(n) = -p(n-2)-p(n-35)+p(n-57)+ \cdots,$$
$$\equiv (n-4)\sigma_7(n)-(5n^2-2n+1)\sigma_5(n)-(5n^3-n^2-n-5)\sigma_3(n)$$
$$-(2n^4+2n^3-2n^2-4n-3)\sigma_1(n); \quad (11.2)$$

$$I_{11m+3}(n) = 0; \quad (11.3)$$

$$I_{11m+4}(n) = -p(n-15)+p(n-26)-p(n-70)+ \cdots,$$
$$\equiv (n-5)\sigma_7(n)-(5n^2+3n-4)\sigma_5(n)-(5n^3-4n^2+4n-1)\sigma_3(n)$$
$$-(2n^4-3n^3+n^2-3n+2)\sigma_1(n); \quad (11.4)$$

$$I_{11m+5}(n) = p(n-5)-p(n-126)-p(n-247)+ \cdots,$$
$$\equiv -5n\sigma_7(n)+3n^2\sigma_5(n)+3n^3\sigma_3(n)-n^4\sigma_1(n); \quad (11.5)$$

$$I_{11m+6}(n) = 0; \quad (11.6)$$

$$I_{11m+7}(n) = p(n-7)-p(n-40)+p(n-51)- \cdots,$$
$$\equiv (n-1)\sigma_7(n)-(5n^2+5n-2)\sigma_5(n)-(5n^3-3n^2+2n-3)\sigma_3(n)$$
$$-(2n^4-5n^3+4n^2+2n-1)\sigma_1(n); \quad (11.7)$$

$$I_{11m+8}(n) = I_{11m+9}(n) = I_{11m+10}(n) = 0. \quad (11.8\text{-}11.10)$$

I have already mentioned that congruences (5.1), (7.2) and (11.5) were obtained by Ramanujan. But it seems that it was not noticed that the expressions occuring therein were I_{am+b}'s, as it appears from the fact that Ramanujan obtained the left hand expression in (11.5) by considering the coefficient of x^n in the expansion of

$$x^5 f(x^{121})/f(x),$$

where

$$f(x) = (1-x)(1-x^2)(1-x^3) \cdots ,$$

6. It may be pointed out that Ramanujan's well-known congruences

$$\left.\begin{aligned} p(5m+4) &\equiv 0 \pmod{5}, \\ p(7m+5) &\equiv 0 \pmod{7}, \\ p(11m+6) &\equiv 0 \pmod{11}. \end{aligned}\right\} \qquad (12)$$

follow by induction from the recursive congruences established above.

$$p(5m+4) \equiv 0 \pmod{5}$$

follow directly from any one of the relations (5.0), (5.1), (5.2). That for modulus 7 follows directly from (7.2). But any one of (7.0), (7.1) and (7.5) may also be used, although in a less direct form, for they require respectively the further, although simple, results

$$\sigma_3(7m+5) \equiv 0 \pmod{7},$$
$$\sigma_3(7m+6) \equiv 0 \pmod{7},$$
$$\sigma_3(7m+3) \equiv 0 \pmod{7}.$$

For modulus 11 the best relation to be used is (11.5), others may also be used less directly.

In his paper Ramanujan (1921) derives his recursive congruences and (12) independently.

I take this opportunity of offering my sincere thanks to Mr. R. C. Bose for his very kind interest in the preparation and publication of this paper.

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A NOTE ON SKEWNESS AND KURTOSIS

By

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1. Let n be an integer ≥ 2. Let $x_1, x_2, x_3, \ldots, x_n$ be n real quantities, not all equal. Let us set

$$\bar{x} = (x_1+x_2+x_3+\ldots+x_n)/n,$$

$$\alpha_3(x_1, x_2, \ldots, x_n) = \left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3\right\}\Big/\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}^{3/2},$$

$$\alpha_4(x_1, x_2, \ldots, x_n) = \left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^4\right\}\Big/\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}^2.$$

It will be observed that α_3 and α_4 are the familiar functions in Statistics used for measuring respectively skewness and kurtosis.

In a recent paper, Wilkins (1944) gives a proof of the well known inequality

$$\alpha_4 \geq 1+\alpha_3^2. \tag{1}$$

He further proves the new result

$$\operatorname{Max} \alpha_3(x_1, x_2, \ldots, x_n) = (n-2)/(n-1)^{\frac{1}{2}}. \tag{2}$$

In the present note, by following an entirely different process, I have obtained the results (1) and (2). Further I have proved

$$\operatorname{Max} \alpha_4(x_1, x_2, \ldots, x_n) = n-2+1/(n-1). \tag{3}$$

It is believed that the above result is new. The note also contains a few interesting properties of the function $\alpha_4(x_1, x_2, \ldots, x_n)$.

2. Let us set

$$y_i = (x_i-\bar{x})\Big/\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}^{\frac{1}{2}}. \tag{4}$$

Then

$$\alpha_3 = \sum_{i=1}^{n} y_i^3/n, \tag{5}$$

$$\alpha_4 = \sum_{i=1}^{n} y_i^4/n \tag{6}$$

where $y_1, y_2, \ldots, y_n$ satisfy the conditions

$$\sum_{i=1}^{n} y_i = 0, \tag{7}$$

$$\sum_{i=1}^{n} y_i^2 = n. \tag{8}$$

3. *Proof of* (1): The real quantities $y_1, y_2, \ldots, y_n$ satisfy the equation

$$0 = f(y) = (y-y_1)(y-y_2)\ldots(y-y_n) = y^n - \frac{n}{2}y^{n-2} - \frac{n\alpha_3}{3}y^{n-3} + \frac{n(n-2\alpha_4)}{8}y^{n-4} + \cdots \quad (9)$$

Consider the chain of Sturm's functions

$$f(y) = y^n - \tfrac{1}{2}ny^{n-2} - \cdots$$

$$\frac{f'(y)}{n} = y^{n-1} - \tfrac{1}{2}(n-2)y^{n-3} + \cdots$$

$$f_2(y) = y^{n-2} + \alpha_3 y^{n-3} + \cdots$$

$$f_3(y) = (\alpha_4 - 1 - \alpha_3^2)y^{n-3} + \cdots$$

etc. etc. etc.

Since all the roots of $f(y) = 0$ are real and since the coefficient of y^n in $f(y)$ is positive, the leading coefficient in each of the above functions must be non-negative. Hence we get (1).*

4. *Proof of* (2): (*i*) For $n = 2$, by (7) $y_1 = -y_2$ and by (8) $y_1^2 = 1$

$$\therefore \; y_1 = -y_2 = \pm 1.$$

Hence $\alpha_3 = 0 =$ the value of the right-hand expression in (2) when $n = 2$.

(*ii*) Let $n \geq 3$. Differentiating (9) $n-3$ times and re-writing the equation so as to make the leading coefficient unity, we get

$$y^3 - \frac{3}{n-1}y - \frac{2\alpha_3}{(n-1)(n-2)} = 0. \quad (10)$$

Since all the roots of (9) are real, all the three roots of (10) are also real. It is well known that if $y^3 + 3b_2x + b_3 = 0$ has all its roots real, then $b_3^2 + 4b_2^3 \leq 0$. In this case $b_2 = -\dfrac{1}{n-1}$ and $b_3 = \dfrac{-2\alpha_3}{(n-1)(n-2)}$ and the condition gives

$$\frac{4\alpha_3^2}{(n-1)^2(n-2)^2} - \frac{4}{(n-1)^3} \leq 0$$

whence we get

$$\alpha_3 \leq (n-2)/(n-1)^{\frac{1}{2}}.$$

Since corresponding to $y_1 = \sqrt{(n-1)}$, $y_i = -1/\sqrt{(n-1)}$, $i = 2, 3, \ldots\ldots, n$ $\alpha_3 = (n-2)/(n-1)^{\frac{1}{2}}$, we get (2).

It may be observed in passing that as $\alpha_3(x_1, x_2, \ldots, x_n)$ is an odd function, $\operatorname{Min} \alpha_3(x_1, x_2, \ldots, x_n) = -(n-2)/(n-1)^{\frac{1}{2}}$ and we can write

$$-(n-2)/(n-1)^{\frac{1}{2}} \leq \alpha_3 \leq (n-2)/(n-1)^{\frac{1}{2}}.$$

5. *The function* α_4: From (1) it is obvious that $\alpha_4(x_1, x_2, \ldots, x_n) \geq 1$. This can be obtained more simply from a consideration of the equation

$$\frac{1}{n}\sum_{i=1}^{n}(y + y_i^2)^2 = y^2 + 2y + \alpha_4 = 0.$$

* The above proof is valid for $n \geq 3$. The case $n = 2$ is trivial, since in this case $\alpha_4 = 1$, $\alpha_3 = 0$ and (1) is obvious.

Since this equation can not have two distinct real roots, it follows that $\alpha_4 \geq 1$. Incidentally we observe that equality will hold if and only if $y_1^2 = y_2^2 = \ldots = y_n^2$. From (8), $y_i^2 = 1$ and therefore $y_i = \pm 1$ $(i = 1, 2, \ldots, n)$. Since these must also satisfy (7) we get the results:

(*i*) $\alpha_4(x_1, x_2, \ldots, x_n) > 1$ if n be odd;

(*ii*) $\alpha_4(x_1, x_2, \ldots, x_n)$ can be equal to 1 when n is even (equal to $2k$, say). This value will be assumed, for example, if

$$x_i = a+\sqrt{b}, \qquad i = 1, 2, \ldots, k,$$
$$x_j = a-\sqrt{b}, \qquad j = k, k+1, \ldots, 2k$$

where a is any real number and $b > 0$.

Since

$$n^2 = \left\{\sum_{i=1}^{n} y_i^2\right\}^2 = \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} y_i^2 y_j^2 > n\alpha_4$$

$y_1 = y_2 = \ldots = y_n = 0$ being impossible on account of (8). If follows, therefore, that $\alpha_4 < n$. Hence we can write

$$1 \leq \alpha_4(x_1, x_2, \ldots, x_n) < n.$$

6. *Proof of* (3): (*i*) Let $n = 2$. From (7), $y_1 = -y_2$ and from (8) $y_1^2 = 1$

$$\therefore \quad y_1 = -y_2 = \pm 1$$

and $\alpha_4 = 1 =$ the value of the right-hand expression in (3) for $n = 2$.

(*ii*) Let $n = 3$. Since by (7) and (8)

$$9 = (y_1^2+y_2^2+y_3^2)^2 = (y_1^4+y_2^4+y_3^4)+2(y_1y_2+y_1y_3+y_2y_3)^2-4y_1y_2y_3(y_1+y_2+y_3) = 3\alpha_4+\tfrac{9}{2}.$$

Therefore $\alpha_4(x_1, x_2, x_3) = \frac{3}{2} =$ the value of the right-hand member in (3) corresponding to $n = 3$.

(*iii*) Let $n \geq 4$. Differentiating (9) $n-4$ times and re-writing the equation so as to make the leading coefficient unity, we get

$$y^4 - \frac{6}{n-1}y^2 + \frac{8\alpha_3}{(n-1)(n-2)}y + \frac{3(n-2\alpha_4)}{(n-1)(n-2)(n-3)} = 0. \tag{11}$$

Since (9) has only real roots, (11) must also have 4 real roots. It is well known that a necessary condition for the reality of all the roots of the quartic $y^4+6b_2y^2+4b_3y+b_4 = 0$ is that the quantity $I = b_4+3b_2^2 \geq 0$. Hence we get

$$\frac{(n-2\alpha_4)}{(n-1)(n-2)(n-3)} + \frac{1}{(n-1)^2} \geq 0$$

whence

$$\alpha_4 \leq (n-2)+1/(n-1).$$

Hence

$$\text{Max}\, \alpha_4(x_1, x_2, \ldots, x_n) \leq (n-2)+1/(n-1).$$

Since for $x_1 = \sqrt{(n-1)}$, $x_i = -1/\sqrt{(n-1)}$, $i = 2, 3, \ldots, n$, $\alpha_4 = (n-2)+1/(n-1)$, (3) follows.

My thanks are due to Dr. N. M. Basu for his kind interest in the preparation of this note.

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ON A FEW GENERALISATIONS OF WEIERSTRASS' NON-DIFFERENTIABLE FUNCTIONS

BY

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Weierstrass' function is

$$f(x) = \sum_{r=0}^{\infty} a^r \cos(b^r \pi x),$$

where b is an odd integer, $0 < a < 1$, and $ab > 1 + 3\pi(1-a)/2$. Weierstrass' result has been generalised in various ways by Hertz (1875), Dini, Darbaux, Lerch, Faber, Landsberg, Bromwich, Young, Hardy, and Mukhopadhyay (1938).

Hertz (1875) has shown that the function

$$\sum_{r=0}^{\infty} a^r \cos^p(b^r \pi x)$$

represents a continuous, but non-differentiable function, where $0 < a < 1$, b is positive odd integer, and p is positive odd integer such that $ab > 1 + 3p\pi/2$.

Here I shall prove that the function

$$f(x) = \sum_{r=0}^{\infty} a^r \cos^p(b^r \pi m x) \qquad (1)$$

represents a continuous but non-differentiable function, where $0 < a < 1$, b and m are positive odd integers and p be even or odd positive integer such that $ab > 1 + 3pm\pi(1-a)/2$. Then I shall construct non-differentiable functions containing Legendre's polynomials of the first kind.

Let x have a fixed value and c_n be the integer corresponding to each value of n, such that $c_n - \frac{1}{2} \leqq b^n x < c_n + \frac{1}{2}$.

First, let $x_2 = (c_n - 1)/b^n$, $x_1 = (c_n + \frac{1}{2})/b^n$, and we have

$$I(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{2}{3} b^n \left[\sum_{r=0}^{n-1} a^r \{\cos^p(b^r \pi m x_1) - \cos^p(b^r \pi m x_2)\} - (-1)^{p(c_n - 1)} \frac{a^n}{1-a} \right].$$

Now

$$|\cos^p(b^r \pi m x_1) - \cos^p(b^r \pi m x_2)| \leqq b^r p \pi m (x_1 - x_2) \leqq \frac{3}{2} \cdot \frac{\pi m p b^r}{b^n},$$

therefore we have

$$I(x_1, x_2) = -\frac{2}{3} a^n b^n \frac{(-1)^{p(c_n - 1)}}{1-a} + \lambda p m \pi \frac{a^n b^n - 1}{ab - 1},$$

where $-1 \leqq \lambda \leqq 1$. Let

$$\frac{2}{3} \cdot \frac{1}{1-a} > \frac{pm\pi}{ab-1}, \quad i.e., \; ab > 1 + \tfrac{3}{2} pm\pi(1-a),$$

then we have

$$I(x_1, x_2) = -(-1)^{p(c_n-1)}a^nb^nN_n - \frac{\lambda m p\pi}{ab-1},$$

where $N_n > 0$.

Similarly if $x_2' = (c_n - \frac{1}{2})/b^n$, $x_1' = (c_n+1)/b^n$, we have

$$I(x_1', x_2') = (-1)^{p(c_n-1)}a^nb^nN_n' - \frac{\lambda'\pi mp}{ab-1},$$

where $N_n' > 0$, and $-1 \leq \lambda' \leq 1$ and $ab > 1 + \frac{3}{2}pm\pi(1-a)$.

Let p be *odd integer*. If $\{c_n\}$ be a sequence of even integers, it is seen that as c_n has successive values in the sequence, $I(x_1, x_2)$ is positive and increases indefinitely in numerical values and $I(x_1', x_2')$ is negative and increases indefinity in numerical valuos. Therefore $f(x)$ has no differential co-efficient at the point x. The same conclusion can be made in case $\{c_n\}$ contains a sequence of odd integers

Secondly, let p be *even integer*. If $\{c_n\}$ be a sequence of even integers, then $I(x_1, x_2)$ is negative and increases indefinitely in numerical value, and $I(x_1', x_2')$ is positive and increases indefinitely in numerical value. Therefore $f(x)$ has no differential co-efficient at the point x. The same conclusion can be made if $\{c_n\}$ contains sequence of odd integers.

Let $P_p(z)$ be the Legendre's polynomial which is equal to

$$\sum_{s=0}^{q} \frac{(-1)^s(2p-2s)!}{2^p\, s!\,(p-s)!\,(p-2s)!} z^{p-2s},$$

where $q = p/2$ or $(p-1)/2$ whichever is an integer. Let

$$f(x) = \sum_{r=0}^{\infty} a^r P_p\{\cos(b^r\pi mx)\} = \sum_{s=0}^{q} \frac{(-1)^s(2p-2s)!}{2^p\, s!\,(p-s)!\,(p-2s)!} \sum_{r=0}^{\infty} a^r \cos^{p-2s}(b^r\pi mx)$$

$=$ sum of finite number of series of the type (1).

Since each of these series is non-differentiable, $f(x)$ is non-differentiable.

We know (Adams, 1878) that

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \frac{A_{m-r}A_rA_{n-r}}{A_{n+m-r}}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}(z).$$

where $A_m = 1.3.5.\ldots(2m-1)/m!$ and $m \leq n$. Then

$$f(x) = \sum_{r=0}^{\infty} a^r P_p\{\cos(b^r\pi mx)\}.P_q\{\cos(b^r\pi mx)\}$$

will be non-differentiable function. Similarly

$$f(x) = \sum_{r=0}^{\infty} a^r P_{p_1}\{\cos(b^r\pi mx)\}.P_{p_2}\{\cos(b^r\pi mx)\}.\ldots P_{p_s}\{\cos(b^r\pi mx)\}$$

will be non-differentiable function if $ab > 1 + \frac{3}{2}pm\pi(1-a)$.

Let

$$f(x) = \sum_{r=1}^{\infty} \frac{a^r}{1^q.3^q.\ldots(2r-1)^q} \cos^p\{1^q.3^q.\ldots(2r-1)^q x\}.$$

Further let x have a fixed value and c_n be an integer corresponding to each value of x such that

$$(c_n-\tfrac{1}{2})\pi \leqq 1^q.3^q.\ldots(2r-1)^q x < (c_n+\tfrac{1}{2})\pi$$

where q is a positive integer. Let

$$x_2 = \frac{(c_n-1)\pi}{1^q.3^q.\ldots(2n-1)^q}, \quad x_1 = \frac{(c_n+\frac{1}{2})\pi}{1^q.3^q.\ldots(2n-1)^q},$$

$$x_2' = \frac{(c_n-\frac{1}{2})\pi}{1^q.3^q.\ldots(2n-1)^q}, \quad x_1' = \frac{(c_n+1)\pi}{1^q.3^q.\ldots(2n-1)^q},$$

and we have

$$I(x_1, x_2) = \frac{f(x_1)-f(x_2)}{x_1-x_2} = \lambda p.\frac{a^n-1}{a-1} - (-1)^{p(c_n-1)}\frac{2\mu}{3\pi}.\frac{a^n}{1-a/(2n)^q},$$

where $-1 \leqq \lambda \leqq 1$, $0<\mu<1$; and

$$I(x_1', x_2') = \lambda' p.\frac{a^n-1}{a-1} + (-1)^{p(c_n-1)}\frac{2\mu}{3\pi}.\frac{a^n}{1-a/(2n)^q},$$

where $-1 \leqq \lambda' \leqq 1$, $0<\mu<1$; and $a>1+\frac{3}{2}p\pi$; then $I(x_1, x_2)$ and $I(x_1', x_2')$ have not the same unique limit whether the sequence $\{c_n\}$ is a sequence of even or odd integers. Hence $f(x)$ is non-differentiable function. Hence

$$f(x) = \sum_{r=1}^{\infty} \frac{a^r}{1^q.3^q.\ldots(2r-1)^q}.P_{p_1}\{\cos[1^q.3^q.\ldots(2r-1)^q x]\}.P_{p_2}\{\cos[1^q.3^q.\ldots(2r-1)^q x]\}\ldots P_{p_r}\{\cos[1^q.3^q.\ldots(2r-1)^q x]\}$$

is a non-differentiable function.

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ON A CERTAIN ARITHMETIC FUNCTION

By

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Let
$$\sigma(n) = \sum_{d/n} d,$$
$$L(n) = \sigma(n) - (1+n)$$
taking $n > 1$, $L(0) = 0$; so that $L(n)$ denotes the sum of the divisors of n excepting unity and itself. Further, it is easy to see that $L(p) = 0$, if p is a prime number. Writing
$$L_1(n) = L(n),$$
$$L_2(n) = L\{L_1(n)\},$$
$$L_3(n) = L\{L_2(n)\},$$
$$\text{etc. etc.,}$$
it appears that for given n and variable r ($r = 1, 2, 3, \ldots$), the sequence
$$L_r(n) \qquad [r = 1, 2, 3, \ldots]$$
assumes only a finite number of values.

This conjecture was suggested to me by Professor S. Chowla, the Government College, Lahore. *I have tested this conjecture upto $n = 100$ and find that it is true upto this limit.* It appears that for most of the positive integers n
$$L_r(n) = 0 \tag{1}$$
for a value of r which is small as compared to n. Thus for $1 \leq n \leq 100$ there is a positive integer r satisfying (1) except in three cases namely $n = 48$, 75 and 92.

In these cases we have
$$L_1(48) = 75, \quad L_2(48) = 48,$$
$$L_1(75) = 48, \quad L_2(75) = 75,$$
$$L_1(92) = 75, \quad L_2(92) = 48,$$
$$L_3(92) = 75, \quad L_4(92) = 75, \quad \text{etc.}$$

Example of the other type is $n = 85$. Here
$$L_1(85) = 22, \quad L_2(85) = 13, \quad L_3(85) = 0.$$

Again, taking $n = 18$, we have
$$L_1(18) = 20, \quad L_2(18) = 21, \quad L_3(18) = 10,$$
$$L_4(18) = 7, \quad L_5(18) = 0, \quad L_6(18) = 0, \quad \text{etc.}$$

Thus $L_r(18)$ assumes only 5 values namely 20, 21, 10, 7, 0.

It so seems to me that given any positive integer A, there always exist n such that the sequence
$$L_r(n) \qquad [r = 1, 2, 3, \ldots]$$
assumes at least A values.

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A NOTE ON BIANCHI CONGRUENCE

By

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It is proved in Eisenhart's Differential Geometry (p. 416) that on the focal surfaces of a Bianchi congruence the lines of curvature correspond and likewise the asymptotic lines.

The object of this note is to show that *on the focal surfaces of a Bianchi congruence the characteristic lines also correspond and to determine the equation of these lines.*

The equation of characteristic lines is given by (Eisenhart, 1909, p. 131)

$$[D(GD-ED'')-2D'(FD-ED')]du^2+2[D'(GD+ED'')-2FDD'']dudv$$
$$+[2D'(GD'-FD'')-D''(GD-ED'')]dv^2=0 \quad (1)$$

where E, F, G ; D, D', D'' are the fundamental quantities for a surface.

But for the two focal surfaces fundamental quantities are given (Eisenhart, pp. 410-411) by (developables of the congruence being parametric curves)

$$E_1=4\left(\frac{\partial\rho}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}'\rho\right)^2, \qquad F_1=-4\rho\begin{Bmatrix}12\\1\end{Bmatrix}'\left(\frac{\partial\rho}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}'\rho\right),$$

$$G_1=4\rho^2\left(\mathcal{G}+\begin{Bmatrix}12\\1\end{Bmatrix}'^2\right), \qquad D_1=-\frac{2\mathcal{K}}{\sqrt{\mathcal{G}}}\left(\frac{\partial\rho}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}'\rho\right),$$

$$D_1'=0, \qquad D_1''=-\frac{2\mathcal{K}}{\sqrt{\mathcal{G}}}\begin{Bmatrix}22\\1\end{Bmatrix}'\rho,$$

and

$$E_2=4\rho^2\left(\mathcal{E}+\begin{Bmatrix}12\\2\end{Bmatrix}'^2\right), \qquad F_2=-4\rho\begin{Bmatrix}12\\2\end{Bmatrix}'\left(\frac{\partial\rho}{\partial v}+\begin{Bmatrix}12\\1\end{Bmatrix}'\rho\right),$$

$$G_2=4\left(\frac{\partial\rho}{\partial v}+\begin{Bmatrix}12\\1\end{Bmatrix}'\rho\right)^2, \qquad D_2=\frac{2\mathcal{K}}{\sqrt{\mathcal{E}}}\begin{Bmatrix}11\\2\end{Bmatrix}'\rho,$$

$$D_2'=0, \qquad D_2''=\frac{2\mathcal{K}}{\sqrt{\mathcal{E}}}\left(\frac{\partial\rho}{\partial v}+\begin{Bmatrix}12\\1\end{Bmatrix}'\rho\right).$$

Hence the equation of characteristic lines on the focal surface S_1 of a Bianchi congruence for which $\rho=$ constant and $\partial\rho/\partial u=\partial\rho/\partial v=0$ becomes

$$\left[\begin{Bmatrix}12\\2\end{Bmatrix}'^2\left(\mathcal{G}+\begin{Bmatrix}12\\1\end{Bmatrix}'^2\right)-\begin{Bmatrix}12\\2\end{Bmatrix}'^3\begin{Bmatrix}22\\1\end{Bmatrix}'\right]du^2+4\begin{Bmatrix}12\\1\end{Bmatrix}'\begin{Bmatrix}12\\2\end{Bmatrix}'^2\begin{Bmatrix}22\\1\end{Bmatrix}'dudv$$
$$+\left[\begin{Bmatrix}12\\2\end{Bmatrix}'^2\begin{Bmatrix}22\\1\end{Bmatrix}'^2-\left(\mathcal{G}+\begin{Bmatrix}12\\1\end{Bmatrix}'^2\right)\begin{Bmatrix}12\\2\end{Bmatrix}'\begin{Bmatrix}22\\1\end{Bmatrix}'\right]dv^2=0. \quad (2)$$

But for this surface (Eisenhart, p. 416)

$$\frac{1}{\mathcal{E}}\begin{Bmatrix}12\\2\end{Bmatrix}' + \frac{1}{\mathcal{G}}\begin{Bmatrix}22\\1\end{Bmatrix}' = 0. \tag{3}$$

Hence the equation (2) becomes with the help of equation (3)

$$\mathcal{E}\left[\mathcal{E}\mathcal{G}+\mathcal{E}\begin{Bmatrix}12\\1\end{Bmatrix}'^2 + \mathcal{G}\begin{Bmatrix}12\\2\end{Bmatrix}'^2\right]du^2 - 4\mathcal{E}\mathcal{G}\begin{Bmatrix}12\\1\end{Bmatrix}'\begin{Bmatrix}12\\2\end{Bmatrix}'dudv$$
$$+\mathcal{G}\left[\mathcal{E}\mathcal{G}+\mathcal{G}\begin{Bmatrix}12\\2\end{Bmatrix}'^2 + \mathcal{E}\begin{Bmatrix}12\\1\end{Bmatrix}'^2\right]dv^2 = 0. \tag{4}$$

Similarly using the relation (Eisenhart, p. 416)

$$\frac{1}{\mathcal{E}}\begin{Bmatrix}11\\2\end{Bmatrix}' + \frac{1}{\mathcal{G}}\begin{Bmatrix}12\\1\end{Bmatrix}' = 0 \tag{5}$$

the equation of characteristic lines on the focal surface S_2 is given by

$$\mathcal{E}\left[\mathcal{E}\begin{Bmatrix}12\\1\end{Bmatrix}'^2 + \mathcal{G}\begin{Bmatrix}12\\2\end{Bmatrix}'^2 + \mathcal{E}\mathcal{G}\right]du^2 - 4\mathcal{E}\mathcal{G}\begin{Bmatrix}12\\1\end{Bmatrix}'\begin{Bmatrix}12\\2\end{Bmatrix}'dudv$$
$$+\mathcal{G}\left[\mathcal{E}\mathcal{G}+\mathcal{G}\begin{Bmatrix}12\\2\end{Bmatrix}'^2 + \mathcal{E}\begin{Bmatrix}12\\1\end{Bmatrix}'^2\right]dv^2 = 0$$

which is the same as equation (4).

Hence on the two focal surfaces of a Bianchi congruence the characteristic lines correspond.

For his kind guidance and valuable help the author is thankful to Dr. Ram Behari.

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ON YOUNG'S MODULUS FOR INDIA RUBBER

By

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It is generally made out that the dynamical value, E, of Young's modulus for India rubber is greater than the statical value E_1 (Deodhar and Kothari, 1928). In some cases it is found that E is almost equal to $2E_1$ (Puri, 1937). Various reasons have been given to explain this disprepancy. We propose to shew that the mistake, or at any rate a good part of it, arises from the use of Hooke's Law which holds good only when the longitudinal stretch is small. For large values of the stretch the theory of Finite Strain (Seth, 1935) developed by us should be used.

To begin with we find the extension produced in an elastic string suspended by one extremity and having a weight W attached to the other extremity.

Let OA_1 be the unstretched string, P_1Q_1 any element of its length. Let OA be the stretched string, PQ be the corresponding portion of P_1Q_1. Let w be the weight of a unit of length of the unstretched string, and let $l_1 = OA_1$, $x_1 = OP_1$; $l = OA$, $x = OP$. Then the tension T at P is given by

$$T = w(l_1 - x_1) + W. \tag{1}$$

If we use Hooke's Law we have the relation

$$T = E_1 \frac{dx - dx_1}{dx_1}, \tag{2}$$

and the extension in the string produced is easily seen to be

$$l - l_1 = \frac{l_1}{E} (\tfrac{1}{2} w l_1 + W). \tag{3}$$

But in experiments conducted with India rubber the extension is not small. In place of (2) we should use

$$2T = E'[1 - (dx_1/dx)^2], \tag{4}$$

which is the result (Seth, 1935) given by the Finite Strain theory. Thus

$$\left(\frac{dx_1}{dx}\right)^2 = 1 - \frac{2}{E'}[w(l_1 - x_1) + W], \tag{5}$$

which gives

$$x = \frac{E'}{w}\left[\left\{1 - \frac{2}{E'}(w\,\overline{l_1 - x_1} + W)\right\}^{\frac{1}{2}} - \left\{1 - \frac{2}{E'}(l_1 w + W)\right\}^{\frac{1}{2}}\right]. \tag{6}$$

The extended length l is therefore given by

$$lw/E' = (1 - 2W/E')^{\frac{1}{2}} - [1 - (2/E')(l_1 w + W)]^{\frac{1}{2}}. \tag{7}$$

If $W = 0$, we get

$$E' = \frac{l^2 w}{2(l - l_1)}. \tag{8.1}$$

Also

$$E_1 = \frac{l_1^2 w}{2(l-l_1)^2} \tag{8.2}$$

which shows that

$$E'/E_1 = l^2/l_1^2, \tag{8.3}$$

and hence $E' > E_1$, as it should be. For $l = \sqrt{2}.l_1$ we get $E' = 2E_1$.

If we can take $w = 0$, as is generally the case in experiments conducted with India rubber, for W is very large compared to w, we get

$$E' = \frac{2l^2 W}{l^2 - l_1^2}, \tag{9.1}$$

$$E_1 = \frac{l_1 W}{l - l_1}, \tag{9.2}$$

and hence

$$\frac{E'}{E_1} = \frac{l}{l_1} \cdot \frac{2l}{l+l_1}, \tag{9.3}$$

which again shows that $E' > E_1$. If $l = 1.5l_1$, we get $E' = 1.8E_1$.

Thus in all cases the value of E' obtained by the Finite Strain theory is greater than E_1, obtained by using the Hooke's Law, and hence E' approximates more to the dynamical value than E_1.

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QUANTUM ELECTRODYNAMICS AND THE INTERACTION OF HYDROGEN-LIKE ATOMS WITH A RADIATION FIELD

By

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1. INTRODUCTION

It is well known that when the higher approximation terms of the perturbation theory are taken into account, the quantum electrodynamics of Heisenberg and Pauli (1929 *a* and *b*) gives an infinite displacement of spectrum lines of hydrogen-like atoms. This difficulty arises from the circumstance that when one attempts to solve the wave equation by expressing the wave function as a series in ascending powers of the charge e of the electron (or in powers of the nuclear charge Ze), then it is found that the second and higher order terms of this series contain divergent integrals.

Dirac has recently given a modified form of quantum electrodynamics (Dirac, 1939, 1942) which differs from that of Heisenberg and Pauli in two essential respects. Firstly, it makes use of a certain limiting process (known as the λ-limiting process), which is taken over from the classical theory, where a limiting process is necessary to express the equations of motion in Hamiltonian form. Secondly, Dirac's quantum electrodynamics uses for the purpose of physical interpretation, both positive energy photons and negative energy photons in the process of second quantisation, whereas Heisenberg and Pauli use only positive energy photons.

In a recent paper (Eliezer, 1945 *a*), I have examined the interaction of a free electron with a radiation field, on the basis of Dirac's quantum electrodynamics, and have shown that the interaction is free from divergence, to any order of approximation in the perturbation theory, provided one takes the particular solution which corresponds to outgoing waves of the electron. I have also applied the theory to investigate the probability of certain multiple processes (Eliezer, 1945 *b*).

In this paper I apply Dirac's quantum electrodynamics to examine the interaction of hydrogen-like atoms with a radiation field. We take the nucleus as fixed, and we obtain the wave function of an electron of charge e which is bound to a nucleus of charge $-Ze$. It will be shown that if one takes the particular solution which corresponds to out-going waves, then the solution is free from divergence, to any order of approximation in the perturbation theory.

2. THE WAVE EQUATION AND ITS SOLUTION

The wave equation is

$$(p_0 - H)\psi = 0, \tag{1}$$

where the Hamiltonian H is given by

$$H = \boldsymbol{\alpha}.(\mathbf{p} - e\mathbf{A}) + m\beta + Ze^2/r, \tag{2}$$

where the four-vector x_μ gives the time-coordinate x_0 and the space-coordinate $\mathbf{x}$ of the electron; p_0, $\mathbf{p}$ are the energy and momentum operators given by $p_0 = i\hbar(\partial/\partial x_0)$, $\mathbf{p} = -i\hbar(\partial/\partial\mathbf{x})$, α, β are the usual Dirac matrices. The units are so chosen that the velocity of light is unity. The scalar product notation

$$(a, b) = a^\mu b_\mu = a_0 b_0 - a_r b_r = a_0 b_0 - \mathbf{a}.\mathbf{b}$$

is used, summation from 1 to 3 being implied when Roman suffixes are repeated, and from 0 to 3 when Greek suffixes are repeated. The vector potential $\mathbf{A}(x)$ is expressed as a Fourier expansion by

$$A_r(x) = (\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1}\sum_{k_0=\pm}\int\left\{\xi_{kr}\,e^{i(k,x)} + \xi^*_{kr}\,e^{-i(k,x)}\right\}\partial k,$$

where

$$\partial k = k_0^{-1}\,dk_1\,dk_2\,dk_3,$$

ξ_{kr} and ξ^*_{kr} are respectively the operators of emission and of absorption of a photon with energy and momentum given by $\hbar k_\mu$, $\sum_{k_0=\pm}$ means a summation over both values $\pm\sqrt{(k_1^2+k_2^2+k_3^2)}$ for k_0, and the ξ's satisfy the commutation relations

$$\left.\begin{aligned} \xi_{kr}\xi_{k's} - \xi_{k's}\xi_{kr} = 0, \qquad \xi^*_{kr}\xi^*_{k's} - \xi^*_{k's}\xi^*_{kr} = 0,\\ \xi^*_{kr}\xi_{k's} - \xi_{k's}\xi^*_{kr} = -\tfrac{1}{2}(g_{rs}+l_r l_s)(k_0+k_0')e^{-i(k,\lambda)}\,\delta(k_1-k_1')\delta(k_2-k_2')\delta(k_3-k_3') \end{aligned}\right\} \tag{3}$$

λ_μ being a small time-like four-vector whose direction is within the future light cone, so that

$$\lambda_0 > 0, \quad \lambda^2 = \lambda_0^2 - \boldsymbol{\lambda}^2 > 0, \tag{4}$$

and the metric $g^{\mu\nu}$ is such that $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, while the other components vanish.

To solve the wave equation, we follow the method of the perturbation theory and treat the radiation field as a perturbation, and try a solution of the form

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots \tag{5}$$

where $\psi_0, \psi_1, \ldots$ are in decreasing order of magnitude. We do not make use of the Born approximation according to which the Coulomb field also is treated as a perturbation, because such an approximation will not be valid for our purposes here, and we must therefore retain the term Ze^2/r in the Hamiltonian of the unperturbed system.

Substituting (2) and (5) in (1), and equating terms of the same order, we obtain

$$\left[p_0 - \alpha.\mathbf{p} - m\beta + \frac{Ze^2}{r}\right]\psi_n = -e(\alpha.\mathbf{A})\psi_{n-1}. \tag{6}$$

Taking $n = 0$, we have

$$\left[p_0 - \alpha.\mathbf{p} - m\beta + \frac{Ze^2}{r}\right]\psi_0 = 0. \tag{7}$$

This is the relativistic wave equation giving the stationary states of the atom, and its solution has been investigated by many authors (Dirac, 1935; Darwin, 1928). We denote

by $u_p \exp(-iE_p x_0/\hbar)$ the normalised wave function of a stationary state in which the energy is E_p.

Suppose that initially there are no photons present, and that the atom is in a stationary state of energy E_m. Then

$$\psi_0 = u_m \exp(-iE_m x_0/\hbar). \tag{8}$$

The first approximation term ψ_1 of the wave function is given by

$$\left[p_0 - \alpha.\mathbf{p} - m\beta + \frac{Ze^2}{r}\right]\psi_1 = -e(\alpha.\mathbf{A})u_m \exp(-iE_m x_0/\hbar),$$

which, when we substitute for $\mathbf{A}$, becomes

$$\left[p_0 - \alpha.\mathbf{p} - m\beta + \frac{Ze^2}{r}\right]\psi_1 = -(\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1} e \sum_{k_0=\pm} \int \alpha_r \xi_{kr} u_m \exp(-iE_m x_0/\hbar)\partial k, \tag{9}$$

where we have made use of the relation

$$\xi^* u_m = 0, \tag{10}$$

which is the condition that there are no photons present in the initial state. We solve for ψ_1 by expanding in terms of the eigen functions of the unperturbed system. If we suppose that

$$\psi_1 = \sum_{k_0=\pm} \sum_p a_{pm} u_p \exp\{i(\hbar k_0 - E_m)x_0/\hbar\},$$

then

$$a_{pm} = (\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1} e \int (E_m - E_p - \hbar k_0)^{-1} \xi_{kr} \partial k \left\{\int u_p^* \alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}} d\mathbf{x}\right\},$$

where $d\mathbf{x} = dx_1 dx_2 dx_3$ and the integration is over all space. Hence we take

$$\psi_1 = -(\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1} e \sum_{k_0=\pm} \sum_p \int (E_m - E_p - \hbar k_0)^{-1} \xi_{kr} \partial k \left\{\int u_p^* \alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}} d\mathbf{x}\right\} u_p \exp\{i(\hbar k_0 - E_m)x_0/\hbar\}. \tag{11}$$

The above solution is valid only when $E_m - E_p - \hbar k_0$ does not vanish. To have a solution which applies when $E_m - E_p - \hbar k_0 = 0$, we replace $(E_m - E_p - \hbar k_0)^{-1}$ in (11) by

$$(E_m - E_p - \hbar k_0)^{-1} - i\pi\delta(E_m - E_p - \hbar k_0),$$

thus taking the solution which corresponds to out-going waves (Dirac, 1935; Eliezer 1945 *a*). Hence, instead of (11), we take

$$\psi_1 = -(\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1} e \sum_{k_0=\pm} \sum_p \int \left\{(E_m - E_p - \hbar k_0)^{-1} - i\pi\delta(E_m - E_p - \hbar k_0)\right\} \xi_{kr} \partial k$$
$$\times \left\{\int u_p^* \alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}} d\mathbf{x}\right\} u_p \exp\{i(\hbar k_0 - E_m)x_0/\hbar\}. \tag{12}$$

When we substitute this value of ψ_1 in the equation

$$\left[p_0 - \alpha.\mathbf{p} - m\beta + \frac{Ze^2}{r}\right]\psi_2 = -e(\alpha\mathbf{A})\psi_1$$

from which ψ_2 is obtained, we see that the right-hand side has two terms, one containing $\xi_{k's}\xi_{kr}$ and the other containing $\xi^*_{k's}\xi_{kr}$. The latter term may be reduced by the use of

the commutation relations (8) and the equation (10), and is then seen to be of zero degree in the ξ's. Thus ψ_2 consists of two parts, one of second degree and the other of zero degree in the ξ's. Similarly, for any n ψ_n consists of a sum of terms of degree $n, n-2, n-4, \ldots$ respectively, in the ξ's. Let $\psi_{n,m}$ refer to that part of ψ_n which is of the mth degree in the ξ's. When n is odd, then m is odd; and when n is even, m also is even.

$\psi_{2,2}$ is given by

$$\left[p_0-\boldsymbol{\alpha}.\mathbf{p}-m\beta+\frac{Ze^2}{r}\right]\psi_{2,2} = -(\tfrac{1}{2}\hbar)^{\frac{1}{2}}(2\pi)^{-1}e\sum_{k'_0=\pm}\alpha_s\xi_{k's}e^{i(k',x)}\psi_1\partial k',$$

and hence

$$\psi_{2,2} = \tfrac{1}{2}\hbar(2\pi)^{-2}e^2\sum_{k'_0=\pm}\sum_{k_0=\pm}\sum_{p,q}\iint\left\{(E_m-E_q-\hbar k_0-\hbar k'_0)^{-1}-i\pi\delta(E_m-E_q-\hbar k_0-\hbar k'_0)\right\}$$
$$\times\left\{(E_m-E_p-\hbar k_0)^{-1}-i\pi\delta(E_m-E_p-\hbar k_0)\right\}\xi_{k's}\xi_{kr}\,\partial k'\partial k$$
$$\times\left\{\int u_q^*\alpha_s u_p e^{-i\mathbf{k}'.\mathbf{x}}\,d\mathbf{x}\right\}\left\{\int u_p^*\alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}}\,d\mathbf{x}\right\}u_q\exp\{i(\hbar k_0+\hbar k'_0-E_m)x_0/\hbar\}. \quad (13)$$

Similarly

$$\psi_{2,0} = \tfrac{1}{2}\hbar(2\pi)^{-2}e^2\sum_{k'_0=\pm}\sum_{k_0=\pm}\sum_{p,q}\iint\left\{(E_m-E_q-\hbar k_0+\hbar k'_0)^{-1}-i\pi\delta(E_m-E_q-\hbar k_0+\hbar k'_0)\right\}$$
$$\times\left\{(E_m-E_p-\hbar k_0)^{-1}-i\pi\delta(E_m-E_p-\hbar k_0)\right\}\xi^*_{k's}\xi_{kr}\,\partial k'\partial k$$
$$\times\left\{\int u_q^*\alpha_s u_p e^{-i\mathbf{k}'.\mathbf{x}}\,d\mathbf{x}\right\}\left\{\int u_p^*\alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}}\,d\mathbf{x}\right\}u_q\exp\{i(\hbar k_0-\hbar k'_0-E_m)x_0/\hbar\}. \quad (14)$$

When we use the commutation relations (8) and the equation (10), the equation (14) becomes

$$\psi_{2,0} = -\tfrac{1}{2}\hbar(2\pi)^{-2}e^2\sum_{k'_0=+}\sum_{k_0=\pm}\sum_{p,q}\iint\left\{(E_m-E_q-\hbar k_0+\hbar k'_0)^{-1}-i\pi\delta(E_m-E_q-\hbar k_0+\hbar k'_0)\right\}$$
$$\times\left\{(E_m-E_p-\hbar k_0)^{-1}-i\pi\delta(E_m-E_p-\hbar k_0)\right\}(g_{rs}+l_rl_s)(k_0+k'_0)e^{-i(k,\lambda)}\delta(\mathbf{k}-\mathbf{k}')\partial k'\partial k$$
$$\times\left\{\int u_q^*\alpha_s u_p e^{-i\mathbf{k}'.\mathbf{x}}\,d\mathbf{x}\right\}\left\{\int u_p^*\alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}}\,d\mathbf{x}\right\}u_q\exp\{i(\hbar k_0-\hbar k'_0-E_m)x_0/\hbar\}. \quad (15)$$

We perform the integration with respect to $\mathbf{k}'$ thus obtaining

$$\psi_{2,0} = -\tfrac{1}{2}\hbar(2\pi)^{-2}e^2\sum_{k_0=\pm}\sum_{p,q}\int\left\{(E_m-E_q)^{-1}-i\pi\delta(E_m-E_q)\right\}\left\{(E_m-E_p-\hbar k_0)^{-1}-i\pi\delta(E_m-E_q-\hbar k_0)\right\}$$
$$\times(g_{rs}+l_rl_s)e^{-i(k,\lambda)}\partial k\left\{\int u_q^*\alpha_s u_p e^{-i\mathbf{k}.\mathbf{x}}d\mathbf{x}\right\}\left\{\int u_p^*\alpha_r u_m e^{-i\mathbf{k}.\mathbf{x}}d\mathbf{x}\right\}u_q\exp(-iE_mx_0/\hbar).$$

We integrate with respect to $\mathbf{k}$ by transforming to polar coordinates so that $\partial k = k_0dk_0d\Omega$. We may take without loss of generality $\boldsymbol{\lambda}=0$, which is justifiable from the condition (4).

The integral with respect to k_0 in (15) is of the form

$$\sum_{k_0=\pm}\int\left\{\frac{1}{a-k_0}-i\pi\delta(a-k_0)\right\}k_0\,e^{-ibk_0}\,dk_0,$$

where a, b are independent of k_0. The integral (16) has the value zero, because

$$\int_{-\infty}^{\infty}\sum(a-k_0)^{-1}k_0\,e^{-ibk_0}\,dk_0=2i\pi a e^{-iab}=\int_{-\infty}^{\infty}\sum i\pi\delta(a-k_0)e^{-ibk_0}\,dk_0,$$

Hence

$$\psi_{2,0}=0. \tag{17}$$

When we proceed to the higher approximations, we see that the integrals that occur in the wave function $\psi_{n,m}$ when m is not equal to n, are of the form (16), and we thus have the general result

$$\psi_{n,m}=0,\quad \text{for}\quad m\neq n, \tag{18}$$

$\psi_{n,n}$ can be written down easily for any n, the expression for it being similar to that of $\psi_{2,2}$ in (13).

8. Discussion

We see then that all divergences in the interaction with a radiation field of an electron, which is bound to a nucleus by a Coulomb field, are eliminated when we use the λ-limiting process and the negative energy photons, and take only those solutions which correspond to out-going waves.* If we use a form of the theory which does not make use of the λ-limiting process and the negative energy photons then the integral with respect to k_0 in (15) has the form

$$\int_{0}^{\infty}(a-k_0)^{-1}k_0\,dk_0,$$

which is divergent at infinity. If the theory makes use of the λ-limiting process but not the negative energy photons, then this integral has the form

$$\int_{0}^{\infty}(a-k_0)^{-1}k_0\cos bk_0\,dk_0,$$

which has the same value as

$$a\int_{0}^{\infty}(a-k_0)^{-1}\cos bk_0\,dk_0=(\cos ab)\int_{ab}^{\infty}\frac{\cos x}{x}dx-(\sin ab)\int_{ab}^{\infty}\frac{\sin x}{x}dx,$$

which is covergent at infinity, but does not remain finite as b tends to zero. For small b, the integral is of the order $\log(ab)$, which tends to infinity as b tends to zero. Hence the λ-limiting process, as well as the negative energy photons are both necessary to secure the elimination of divergence.

The above solution shows that radiation damping does not cause any displacement of spectrum lines, nor does it have any effect on other radiative processes such as the

* After the completion of this paper I have been informed by Professor P. A. M. Dirac that he has found a simple general method for handling any one-electron problem. The details are not yet available.

photoelectric effect, bremsstrahlung, etc. The calculations which have already been made to obtain the probabilities of these processes are therefore exact, contrary to what has been believed so far.

Summary

The solution of the wave equation of a hydrogen-like atom interacting with a radiation field is investigated. It is shown that when the radiation interaction is taken as a perturbation, the wave function is free from divergent integrals, to any order of approximation in the perturbation theory, provided one uses the λ-limiting process and the negative energy photons employed by Dirac in his quantum electrodynamics, and also takes the particular solution which corresponds to out-going waves. If the wave function ψ is expressed as a series $\psi_0+\psi_1+\psi_2+\cdots$, where $\psi_0, \psi_1, \ldots$ are in decreasing order of magnitude, and if $\psi_{n,m}$ denotes that part of ψ_n which refers to m photons, then it is shown that $\psi_{n,m}$ is zero for all values of m different from n. It is thus shown that radiation damping does not cause a displacement of spectrum lines, nor does it influence other radiative processes such as the photoelectric effect and bremsstrahlung.

The author records with pleasure his warmest thanks to Professor P. A. M. Dirac for encouragement and advice.

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ON THE CESÀRO SUMMABILITY OF THE SUCCESSIVELY DERIVED SERIES OF THE CONJUGATE SERIES OF A FOURIER SERIES

By

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1. Let $f(\theta)$ be a function which is integrable (L) in $(-\pi, \pi)$ and defined outside this interval by periodicity. Let the Fourier series of $f(\theta)$ be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Then the conjugate series of this Fourier series is

$$\sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta).$$

The series

$$\sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} (a_n \cos n\theta + b_n \sin n\theta), \quad \sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} (b_n \cos n\theta - a_n \sin n\theta)$$

are respectively the rth derived series of the Fourier series and its conjugate series.

Suppose also that there exists a polynomial of $(r-1)$-th degree

$$P(t) = \sum_{p=0}^{r-1} \theta_p \frac{t^p}{p!},$$

such that

$$g(t) = [\{f(\theta+t)-P(t)\}+(-1)^r\{f(\theta-t)-P(-t)\}]/2t^r, \tag{1.1}$$

$$h(t) = [\{f(\theta+t)-P(t)\}-(-1)^r\{f(\theta-t)-P(-t)\}]/2t^r \tag{1.2}$$

are integrable in $(-\pi, \pi)$ and defined by periodicity outside this range. The rth generalised derivative of $f(\theta)$ has been defined to be the limit (Zygmund, 1935)

$$\lim_{t\to 0} r!\, g(t).$$

Wang (1934) has proved the following theorem for the rth derived series of a Fourier series:—

Theorem W. *If $g(t)$ is summable in $(-\pi, \pi)$ and the Fourier series of $g(t)$ is summable (C, δ), $\delta > 0$, to the sum A at $t = 0$, then the* rth *derived series of the Fourier series of $f(\theta)$ is summable $(C, r+\delta)$ to the sum $r!.A$ at the point θ.*

The object of this paper is to investigate the analogous problem for the rth derived series of the conjugate series corresponding to $f(\theta)$. We prove

Theorem 1. *If $h(t)$ is integrable* (L) *in $(-\pi, \pi)$ and is defined by periodicity outside this interval and if the conjugate series corresponding to $h(t)$ is summable (C, α), $\alpha > 0$, to*

the sum S at $t=0$, then the rth derived series of the conjugate series corresponding to $f(\theta)$ is summable $(C, r+\alpha)$ to the sum $r!.S$ at the point θ.

From Theorem 1 and the well-known theorems on the Cesàro summability of the conjugate series, we deduce a number of theorems concerning the Cesàro summability of the rth derived series of the conjugate series.

I am much indebted to Dr. B. N. Prasad for his kind interest and advice.

2. We shall make use of the functions $\gamma_{1+k}(t)$ and $\sigma_{1+k}(t)$ where

$$\gamma_{1+k}(t) = \int_0^1 (1-u)^k \cos tu\, du, \quad \sigma_{1+k}(t) = \int_0^1 (1-u)^k \sin tu\, du.$$

It is well-known that

$$\gamma_{1+k}(t) = \frac{\Gamma(k+1)}{t^{1+k}} C_{1+k}(t), \ k>0, \tag{2.1}$$

where $C_{1+k}(t)$ is Young's function which has the convenient property

$$(d/dt)C_{1+k}(t) = C_k(t). \tag{2.2}$$

Also (Verblunsky, 1932, pp. 398-399)

$$\sigma_{1+k}(t) = \frac{t\gamma_{2+k}(t)}{1+k} = \frac{1-k\gamma_k(t)}{t}.$$

The function $\sigma_{1+k}(t)$ and its derivatives are uniformly bounded for all t and as $t \to \infty$,

$$\sigma_{1+k}(t) = O(1/t), \ k \geq 0,$$
$$\sigma_{1+k}^{(r)}(t) = O(1/t^{r+1}), \ k \geq r. \tag{2.3}$$

We require the lemmas:—

LEMMA 1. $\sigma_{1+k}^{(r)}(t) = \dfrac{1}{t^r}\sum_{p=0}^{r}(-1)^p\, {}^rC_p \dfrac{\Gamma(1+k+p)}{\Gamma(1+k-r+p)}\sigma_{1+k-r+p}(t)$, *for* $k \geq r$.

This is known (Bosanquet, 1934, p. 19) but for the sake of completeness, we prove it here. We have

$$\sigma_{1+k}(t) = \frac{t\gamma_{2+k}(t)}{1+k} = \Gamma(k+1)\frac{C_{2+k}(t)}{t^{1+k}}. \tag{2.4}$$

If r is any integer not greater than k, then differentiating (2.4) r times by Leibniz's theorem and making use of (2.2), we have

$$\sigma_{1+k}^{(r)}(t) = \frac{1}{t^r}\sum_{p=0}^{r}(-1)^p\, {}^rC_p \frac{\Gamma(1+k+p)}{\Gamma(1+k-r+p)} \cdot \frac{C_{2+k-r+p}(t)}{t^{1+k-r+p}},$$

whence the result follows.

LEMMA 2. $$\sum_{p=0}^{r}(-1)^p\, {}^rC_p \frac{\Gamma(1+k+p)}{\Gamma(1+k-r+p)} = (-1)^r r!.$$

We have

$$\sum_{p=0}^{r}(-1)^p\, {}^rC_p\, x^{k+p} = x^k \sum_{p=0}^{r}(-1)^p\, {}^rC_p\, x^p = x^k(1-x)^r. \tag{2.5}$$

Differentiating (2.5) r times and putting $x=1$, we get the result.

3. As shown by Prasad (1931, p. 277) and Verblunsky (1932, p. 400), the kth Rieszian mean of the conjugate series, say $B_0^k(w)$ is given by

$$B_0^k(w) = \frac{w}{\pi}\int_0^\infty \{f(\theta+t)-f(\theta-t)\}\sigma_{1+k}(wt)dt = \frac{w}{\pi}\int_{-\infty}^\infty f(\theta+t)\sigma_{1+k}(wt)dt = \frac{w}{\pi}\int_{-\infty}^\infty f(t)\sigma_{1+k}(\overline{wt-\theta})dt.$$

Similarly the kth Rieszian mean of the rth derived series of the conjugate series is given by

$$B_r^k(w) = \frac{w}{\pi}\int_{-\infty}^\infty f(t)\frac{\partial^r}{\partial\theta^r}[\sigma_{1+k}(\overline{wt-\theta})]dt = (-1)^r\frac{w}{\pi}\int_{-\infty}^\infty f(t)\frac{\partial^r}{\partial t^r}[\sigma_{1+k}(\overline{wt-1})]dt$$

$$= (-1)^r\frac{w}{\pi}\int_{-\infty}^\infty f(\theta+t)\frac{d^r}{dt^r}\sigma_{1+k}(wt)dt = (-1)^r\frac{w^{r+1}}{\pi}\int_{-\infty}^\infty f(\theta+t)\sigma_{1+k}^{(r)}(wt)dt.$$

Now since $\sigma_{1+k}^{(r)}(wt)$ is odd or even according as r is even or odd and since for $t = 0$

$$\sigma_{1+k}^{(r)}(wt) = 0$$

when r is even, we have, by integration by parts and using (2.3), if $r-p$ be odd,

$$w^{r+1}\int_0^\infty t^p\sigma_{1+k}^{(r)}(wt)dt = w^{r+1}\Big[\frac{t^p}{w}\sigma_{1+k}^{(r-1)}(wt) - \frac{pt^{p-1}}{w^2}\sigma_{1+k}^{(r-2)}(wt) + \cdots$$

$$+(-1)^{p-1}\frac{p\,!\,t}{w^p}\sigma_{1+k}^{(r-p)}(wt) + (-1)^p\frac{w^{r+1}}{w^p}\int_0^\infty \sigma_{1+k}^{(r-p)}(wt)dt$$

$$= \Big[O\Big(\frac{1}{t^{r-p}}\Big)\Big]_\infty - w^{r+1-p}\Big[\frac{1}{w}\sigma_{1+k}^{(r-p-1)}(wt)\Big]_0^\infty = O\Big(\frac{1}{t^{r-p}}\Big), \quad \text{as } t\to\infty,$$

$$= 0,$$

for $p = 0, 1, 2, \ldots, (r-1)$. Hence

$$w^{r+1}\int_{-\infty}^\infty t^p\sigma_{1+k}^{(r)}(wt)dt = 0,$$

for $p = 0, 1, 2, \ldots, (r-1)$. Therefore

$$B_r^k(w) = (-1)^r\frac{w^{r+1}}{\pi}\int_{-\infty}^\infty \{f(\theta+t)-P(t)\}\sigma_{1+k}^{(r)}(wt)dt$$

$$= (-1)^r\frac{w^{r+1}}{\pi}\int_0^\infty [\{f(\theta+t)-P(t)\}-(-1)^r\{f(\theta-t)-P(-t)\}]\sigma_{1+k}^{(r)}(wt)dt$$

$$= (-1)^r\frac{2}{\pi}w^{r+1}\int_0^\infty h(t)t^r\sigma_{1+k}^{(r)}(wt)dt,$$

by (1.2). Or

$$B_r^k(w) = (-1)^r\sum_{p=0}^r(-1)^p\,{}^rC_p\frac{\Gamma(1+k+p)}{\Gamma(1+k-r+p)}\cdot\frac{2w}{\pi}\int_0^\infty h(t)\sigma_{1+k-r+p}(wt)dt,$$

by the Lemma 1.

If we put $k = r+\alpha$, this becomes

$$B_r^k(w) = (-1)^r \sum_{p=0}^{r} (-1)^p\, {}^rC_p \frac{\Gamma(1+r+\alpha+p)}{\Gamma(1+\alpha+p)} \cdot \frac{2w}{\pi} \int_0^\infty h(t)\sigma_{1+\alpha+p}(wt)dt$$

$$= (-1)^r \sum_{p=0}^{r} (-1)^p\, {}^rC_p \frac{\Gamma(1+r+\alpha+p)}{\Gamma(1+\alpha+p)} \mathcal{B}_0^{\alpha+p}(wt), \tag{3.1}$$

where $\mathcal{B}_0^{\alpha+p}(w)$ is the $(\alpha+p)$-th Rieszian mean of the conjugate series corresponding to $h(t)$.

Now if the conjugate series corresponding to $h(t)$ is summable (C, α) to S at $t = 0$, then

$$\lim_{w\to\infty} \mathcal{B}_0^{\alpha}(w) = S,$$

and hence by the summability theory,

$$\lim_{w\to\infty} \mathcal{B}^{\alpha+p}(w) = S,$$

for $p = 1, 2, 3, \ldots, r$. Therefore

$$B_r^{r+\alpha}(w) = (-1)^r\left[\sum_{p=0}^{r} (-1)^p\, {}^rC_p \frac{\Gamma(1+r+\alpha+p)}{\Gamma(1+\alpha+p)}\right] S = r!.S,$$

for $\alpha > 0$, by the Lemma 2.

This proves the Theorem 1.

4. Now we shall deduce a number of theorems from the Theorem 1 and some well-known theorems on the Cesàro summability of the conjugate series. In the statement of these theorems we suppose that a polynomial $P(t)$ of $(r-1)$th degree exists such that

$$h(t) = [\{f(\theta+t)-P(t)\}-(-1)^r\{f(\theta-t)-P(-t)\}]/2t^r$$

is periodic and integrable (L) in $(-\pi, \pi)$. Thus we get the following theorems:—

By the application of Hardy and Littlewood's theorem (1926, p. 215)

Theorem 2. *The* rth *derived series of the conjugate series corresponding to* $f(\theta)$ *is summable* $(C, k \geq r)$ *to the integral*

$$\frac{2r!}{\pi} \int_0^\infty \frac{h(t)}{t} dt,$$

provided that this integral exists in Cauchy's sense and provided further that

$$\int_0^t |h(t)|dt = o(t),$$

or in particular

$$h(t) = o(1).$$

Sayer's theorem (Sayer, 1930, p. 39) for the first derived series of the conjugate series is the case $r = 1$ of the particular part of the Theorem 2.

By the application of Prasad's theorem (Prasad, 1931, p. 274), we get

Theorem 3. *The rth derived series of the conjugate series for $f(\theta)$ is summable $(C, k > r)$ to the integral*

$$\frac{2r!}{\pi}\int_0^\infty \frac{h(t)}{t}\,dt,$$

provided that this integral exists in Cauchy's sense and provided further that $h(t)$ is bounded

By the application of a special case of Hardy and Littlewood's (1940, p. 279) theorem, we get

Theorem 4. *The rth derived series of the conjugate series for $f(\theta)$ is summable $(C, k > r)$ to the integral*

$$\frac{2r!}{\pi}\int_0^\infty \frac{h(t)}{t}\,dt,$$

provided that this integral exists in Cauchy's sense and provided further that

$$\int_0^t |h(t)|\,dt = O(t).$$

By the application of Paley's theorem (Paley, 1930), we get

Theorem 5. *If the conj. $\lim_{t\to 0}$ $h(t) = S(C', \alpha)$, $\alpha \geq 0$, then the rth derived series o the conjugate series for $f(\theta)$ is summable $(C, \alpha+r+\delta)$ to $r!.S$, $\delta > 0$*

A theorem, depending upon the use of a generalised type of integrals, has been given by Bosanquet (1940, p. 287) for the summability $(C, \alpha+r)$, $\alpha \geq 0$ and r integer, of the rth derived series of the conjugate series, but the results are of rather complicated character.

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ON INTERPOLATION FORMULAE IN TWO VARIABLES WITH RECIPROCAL DIFFERENCES

By

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A general formulae for interpolation in two variables had been worked out by Dasgupta and Lal (1940). The theory of Reciprocal Differences was introduced by Thiele (1909). Some applications of Reciprocal Differences as applied to two variables had been discussed in two papers [Dasgupta and Singh (1944); Prosad, Singh and Dasgupta (1946)]. The present paper gives further work on the subject.

Defining the operations ρ and σ as

$$\rho f(x, y) = \frac{\hat{w}}{f(x+\hat{w}, y) - f(x, y)} \text{ and } \sigma f(x, y) = \frac{\hat{w}}{f(x, y+\hat{w}) - f(x, y)}$$

and denoting briefly $f(x, y)$ by 00, $f(x+\hat{w}, y)$ by 10, $f(x, y+\hat{w})$ by 01, and so on, and putting

$$\begin{aligned} a &= 10-00, & a' &= 01-00,\\ b &= 11-01, & b' &= 11-10,\\ c &= 20-10, & c' &= 02-01,\\ d &= 21-11, & d' &= 12-11,\\ e &= 12-02, & e' &= 21-20,\\ f &= 22-12, & f' &= 22-21, \end{aligned}$$

we have

$$\rho f(x, y) = \frac{\hat{w}}{f(x+\hat{w}, y) - f(x, y)} = \frac{\hat{w}}{10-00}$$

and

$$\sigma f(x, y) = \frac{\hat{w}}{f(x, y+\hat{w}) - f(x, y)} = \frac{\hat{w}}{01-00}.$$

Now

$$\rho\sigma f(x, y) = \frac{\hat{w}}{\dfrac{\hat{w}}{11-10} - \dfrac{\hat{w}}{01-00}} = \frac{(01-00)(11-10)}{01+10-00-11},$$

$$\sigma\rho f(x, y) = \frac{\hat{w}}{\dfrac{\hat{w}}{11-01} - \dfrac{\hat{w}}{10-00}} = \frac{(10-00)(11-01)}{01+10-00-11}.$$

Denoting, more briefly, $\rho\sigma f(x, y)$ by $\rho\sigma$, and $\sigma\rho f(x, y)$ by $\sigma\rho$, we have,

$$\sigma\rho = ab/(a-b), \tag{1}$$

and

$$\rho\sigma = a'b'/(a'-b'). \tag{2}$$

We note that $a-b$ is an invariant, and so

$$\rho\sigma = \sigma\rho \tag{3}$$

only if ab is also an invariant. Again,

$$\rho\rho = \frac{\hat{w}}{\dfrac{\hat{w}}{20-10}-\dfrac{\hat{w}}{10-00}} = \frac{(20-10)(10-00)}{2.10-00-20} = \frac{ac}{a-c}. \tag{4}$$

Similarly

$$\sigma\sigma = a'c'/(a'-c'). \tag{5}$$

Hence

$$\rho\rho = \sigma\sigma \tag{6}$$

if $ac/(a-c) = a'c'/(a'-c')$, i.e., if $1/a-1/c$ is an invariant.

We find further

$$\rho\sigma\rho = \frac{\hat{w}(10+01-00-11)(20+11-10-21)}{(21-11)(20-10)(10+01-00-11)-(11-01)(10-00)(20+11-10-21)}$$

$$= \frac{\hat{w}(a-b)(c-d)}{cd(a-b)-ab(c-d)}. \tag{7}$$

Similarly

$$\sigma\rho\sigma = \frac{\hat{w}(a'-b')(c'-d')}{c'd'(a'-b')-a'b'(c'-d')}. \tag{8}$$

Since $a-b \equiv a'-b'$, we have

$$\rho\sigma\rho = \sigma\rho\sigma \tag{9}$$

if $a-b \equiv a'-b' = 0$, or if $ab/(a-b)-cd/(c-d)$ is an invariant.

Again we have

$$\sigma\rho\rho = \frac{\hat{w}(2.11-01-21)(2.10-00-20)}{(21-11)(11-01)(2.10-00-20)-(20-10)(10-00)(2.11-01-21)}$$

$$= \frac{\hat{w}(a-c)(b-d)}{bd(a-c)-ac(b-d)}, \tag{10}$$

and

$$\rho\sigma\sigma = \frac{\hat{w}(a'-c')(b'-d')}{b'd'(a'-c')-a'c'(b'-d')}, \tag{11}$$

so

$$\sigma\rho\rho = \rho\sigma\sigma \tag{12}$$

if $ac/(a-c)-bd/(b-d)$ is an invariant.

We have

$$\sigma\sigma\rho = \frac{\hat{w}(11+02-12-01)(10+01-11-00)}{(11-01)(12-02)(10+01-11-00)-(10-00)(11-01)(11+02-12-01)}$$

$$= \frac{\hat{w}(a-b)(b-e)}{be(a-b)-ab(b-e)}. \tag{13}$$

Similarly

$$\rho\rho\sigma = \frac{\hat{w}(a'-b')(b'-e')}{b'e'(a'-b')-a'b'(b'-e')}. \tag{14}$$

Hence

$$\sigma\sigma\rho = \rho\rho\sigma \tag{15}$$

if $a-b=0$, or if $ab/(a-b)-be/(b-e)$ is an invariant.

Again we have

$$\rho\sigma\sigma\rho = 1\Big/\Big[\frac{(21+12-22-11)(20+11-21-10)}{(21-11)(22-12)(20+11-21-10)-(20-10)(21-11)(21+12-22-11)}$$

$$-\frac{(11+02-12-01)(01+10-11-00)}{(11-01)(12-02)(01+10-11-00)-(10-00)(11-01)(11+02-12-01)}\Big]$$

$$=1\Big/\Big[\frac{(c-d)(d-f)}{df(c-d)-cd(d-f)}-\frac{(a-b)(b-e)}{be(a-b)-ab(b-e)}\Big]. \tag{16}$$

Similarly,

$$\sigma\rho\rho\sigma = 1\Big/\Big[\frac{(c'-d')(d'-f')}{d'f'(c'-d')-c'd'(d'-f')}-\frac{(a'-b')(b'-e')}{b'e'(a'-b')-a'b'(b'-e')}\Big]. \tag{17}$$

Particular cases: (α) $a-b\equiv a'-b'=0$, then

$$\rho\sigma\sigma\rho-\sigma\rho\rho\sigma = \frac{df(c-d)-cd(d-f)}{(c-d)(d-f)}-\frac{d'f'(c'-d')-c'd'(d'-f')}{(c'-d')(d'-f')}.$$

So, if $a-b\equiv a'-b'=0$, and if further $cd/(c-d)-df/(d-f)$ is an invariant, then

$$\rho\sigma\sigma\rho = \sigma\rho\rho\sigma. \tag{18}$$

(β) $b-e\equiv c'-d'=0$, then

$$\rho\sigma\sigma\rho-\sigma\rho\rho\sigma = \frac{df(c-d)-cd(d-f)}{(c-d)(d-f)}+\frac{b'e'(a'-b')-a'b'(b'-e')}{(a'-b')(b'-e')}$$

$$=\frac{(a'-b')[df(c-d)-cd(d-f)]+(d-f)[b'e'(a'-b')-a'b'(b'-e')]}{(c-d)(d-f)(a'-b')}.$$

So, if further

$$(a'-b')[df(c-d)-cd(d-f)]+(d-f)[b'e'(a'-b')-a'b'(b'-e')]=0,$$

then $\rho\sigma\sigma\rho=\sigma\rho\rho\sigma$. Now, since $a-b\equiv a'-b'$, the above condition may be written thus:

If $(a-b)[df(c-d)-cd(d-f)]+(d'-f')[b'e'(a'-b')-a'b'(b'-e')]=0$, and $b-e\equiv c'-d'=0$, then

$$\rho\sigma\sigma\rho = \sigma\rho\rho\sigma. \tag{19}$$

(γ) *If $b'-e'\equiv c'-d'=0$, and if further*

$$(a'-b')[d'f'(c'-d')-c'd'(d'-f')]+(d-f)[be(a-b)-ab(b-e)]=0,$$

then it follows, by symmetry from the preceding case, *that*

$$\rho\sigma\sigma\rho = \sigma\rho\rho\sigma. \tag{20}$$

We find again

$$\sigma\rho\sigma\rho = 1\Big/\Big[\frac{(11+02-01-12)(21+12-11-22)}{(22-12)(21-11)(11+02-01-12)-(12-02)(11-01)(21+12-11-22)}$$

$$-\frac{(10+01-11-00)(20+11-10-21)}{(21-11)(20-10)(10+01-00-11)-(11-01)(10-00)(20+11-10-21)}\Big]$$

$$= 1\Big/\Big[\frac{(b-e)(d-f)}{df(b-e)-be(d-f)}-\frac{(a-b)(c-d)}{cd(a-b)-ab(c-d)}\Big]. \tag{21}$$

Similarly,

$$\rho\sigma\rho\sigma = 1\Big/\Big[\frac{(b'-e')(d'-f')}{d'f'(b'-e')-b'e'(d'-f')}-\frac{(a'-b')(c'-d')}{c'd'(a'-b')-a'b'(c'-d')}\Big]. \tag{22}$$

Particular cases: (α) $a-b \equiv a'-b' = 0$,

$$\sigma\rho\sigma\rho-\rho\sigma\rho\sigma = \frac{df(b-e)-be(d-f)}{(b-e)(d-f)}-\frac{d'f'(b'-e')-b'e'(d'-f')}{(b'-e')(d'-f')}.$$

So, if $a-b \equiv a'-b' = 0$, and if further $be/(b-e)-df/(d-f)$ is an invariant, then

$$\sigma\rho\sigma\rho = \rho\sigma\rho\sigma. \tag{23}$$

(β) If $c-d \equiv b'-e' = 0$,

$$\sigma\rho\sigma\rho-\rho\sigma\rho\sigma = \frac{df(b-e)-be(d-f)}{(b-e)(d-f)}+\frac{c'd'(a'-b')-a'b'(c'-d')}{(a'-b')(c'-d')}$$

$$= \frac{(a'-b')[df(b-e)-be(d-f)]+(d-f)[c'd'(a'-b')-a'b'(c'-d')]}{(a'-b')(b-e)(d-f)}$$

(since $c'-d' = b-e$). So, if further

$$(a'-b')[df(b-e)-be(d-f)]+(d-f)[c'd'(a'-b')-a'b'(c'-d')] = 0,$$

$\sigma\rho\sigma\rho = \rho\sigma\rho\sigma$. Now, since $a-b \equiv a'-b'$, and $d-f \equiv d'-f'$, the above condition may be written thus:

If $c-d \equiv b'-e' = 0$, and if further $(a-b)[df(b-e)-be(d-f)]+(d'-f')[c'd'(a'-b')-a'b'(c'-d')] = 0$, then

$$\sigma\rho\sigma\rho = \rho\sigma\rho\sigma. \tag{24}$$

(γ) *If $c'-d' \equiv b-e = 0$, and if further*

$$(a'-b')[d'f'(b'-e')-b'e'(d'-f')]+(d-f)[cd(a-b)-ab(c-d)] = 0,$$

it follows from the preceding case by symmetry *that*

$$\sigma\rho\sigma\rho = \rho\sigma\rho\sigma. \tag{25}$$

(δ) If $d-f \equiv d'-f' = 0$, then

$$\sigma\rho\sigma\rho-\rho\sigma\rho\sigma = \frac{ab(c-d)-cd(a-b)}{(a-b)(c-d)}-\frac{a'b'(c'-d')-c'd'(a'-b')}{(a'-b')(c'-d')}.$$

So, if $d-f \equiv d'-f' = 0$, and if further $ab/(a-b)-cd/(c-d)$ is an invariant, then

$$\sigma\rho\sigma\rho = \rho\sigma\rho\sigma \tag{26}$$

Again we have,

$$\sigma\sigma\rho\rho = 1\Big/\Big[\frac{(2.12-02-22)(2.11-01-21)}{(22-12)(12-02)(2.11-01-21)-(21-11)(11-01)(2.12-02-22)}$$

$$-\frac{(2.11-01-21)(2.10-00-20)}{(21-11)(11-01)(2.10-00-20)-(20-10)(10-00)(2.11-01-21)}\Big]$$

$$= 1\Big/\Big[\frac{(b-d)(e-f)}{ef(b-d)-bd(e-f)}-\frac{(a-c)(b-d)}{bd(a-c)-ac(b-d)}\Big]. \tag{27}$$

Similarly,

$$\rho\rho\sigma\sigma = 1\Big/\Big[\frac{(b'-d')(e'-f')}{e'f'(b'-d')-b'd'(e'-f')}-\frac{(a'-c')(b'-d')}{b'd'(a'-c')-a'c'(b'-d')}\Big]. \tag{28}$$

Particular cases: (α) If $a-c = 0 = a'-c'$, then

$$\sigma\sigma\rho\rho-\rho\rho\sigma\sigma = \frac{ef(b-d)-bd(e-f)}{(b-d)(e-f)}-\frac{e'f'(b'-d')-b'd'(e'-f')}{(b'-d')(e'-f')}.$$

So, if $a-c \equiv a'-c' = 0$, *and if further* $bd/(b-d)-ef/(e-f)$ *is an invariant, then*

$$\sigma\sigma\rho\rho = \rho\rho\sigma\sigma. \tag{29}$$

(β) If $e-f \equiv e'-f' = 0$, then

$$\sigma\sigma\rho\rho-\rho\rho\sigma\sigma = \frac{ac(b-d)-bd(a-c)}{(a-c)(b-d)}-\frac{a'c'(b'-d')-b'd'(a'-c')}{(a'-c')(b'-d')}.$$

So, if $e-f \equiv e'-f' = 0$, *and if further* $ac/(a-c)-bd/(b-d)$ *is an invariant (in which case* $\rho\sigma\sigma = \sigma\rho\rho$, *as observed in* (12)), *then*

$$\sigma\sigma\rho\rho = \rho\rho\sigma\sigma. \tag{30}$$

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ON PARALLELISM IN RIEMANNIAN SPACE—III

By

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1. This paper is a continuation of a previous paper (Sen, 1945) in which we considered, in a Riemannian space of n dimensions with the fundamental metric tensor g_{ij} and the fundamental vectors ${}_Th^i$ forming an orthogonal ennuple, the parallelism

$$dV^l + \nabla^l_{ij} V^i dx^j = 0 \tag{1.1}$$

where

$$\nabla^l_{ij} = \tfrac{1}{2}\sum_T {}_Th^l\left(\frac{\partial\, {}^Th_i}{\partial x^j} + \frac{\partial\, {}^Th_j}{\partial x^i}\right), \quad g_{ij} = \sum_T {}^Th_i {}^Th_j, \quad {}^Th_i = g_{ij}\, {}_Th^j.$$

With reference to the letters used as indices, small letters are used for coordinate indices while capital letters are used for indices referring to the orthogonal ennuple. The signs of summation with respect to coordinate indices, when they occur once above and once below, are left out, but those with respect to indices referring to the ennuple are retained. Further, in what follows, the notation ( ) with a subscript is used to denote covariant derivative with respect to parallelism (1.1) while the notation { } with a subscript stands for covariant derivative with respect to Levi-Civita parallelism.

Although the process of covariant differentiation is applicable in a space bearing affine connection, the theory was originally based on the concept of parallel displacement. By covariant differentiation with respect to a parallelism is meant here exactly the same process as is usually applied in a space with affine connection, whereby the covariant differentiation of the sum, difference, outer and inner product of tensors obeys the same rules as ordinary differentiation (Eisenhart, 1927). The analytical theory of affine space based on intrinsic differential invariants of this space may be carried over to metric space.

A displacement, for which the covariant differential of a vector is zero, is a parallel displacement of the vector (Struik, 1934), and in a metric space, where we can speak of contravariant and covariant components of the same vector, a parallel displacement of a vector may be given by the vanishing of the covariant differentials of its contravariant components. When the scalar product of two arbitrary vectors remains invariant for a parallel displacement of the vectors, the covariant differentials of both the contravariant and covariant components of a vector vanish for this parallel displacement. The word displacement is of course used in the sense of infinitesimal transportation in the direction of progress.

In the paper referred to it was seen that

$$T^l_{ij} = \nabla^l_{ij} - \left\{ {l \atop ij} \right\} = g^{lt}(g_{ij})_t, \tag{1.2}$$

where $\left\{ {l \atop ij} \right\}$ is the Christoffel symbol.

Now, if we put

$$f_T = (g_{ij})_k V^i dx^j {}_Th^k,$$

then the components f_T of the vector f measures the rate of change, with respect to the arc, of the scalar product of the vectors V and dx when these vectors are given the parallel displacement (1.1) in the direction of ${}_Th$. Since the components of f referred to the local system (the orthogonal ennuple) are f_T, its components referred to the general system are

$$f^i = \sum_T f_T\, {}_Th^i.$$

If therefore the notations d and δ be used to denote increments in connection with the parallelism (1.1) and the Levi-Civita parallelism respectively, it follows from (1.2) that

$$\delta V^i = dV^i + f^i, \quad \delta V_i = dV_i + f_i.$$

Hence the increment of a vector V when the vector is given the Levi-Civita parallel transport along an elementary path dx is equal to the sum of its increment when given the parallel transport (1.1) along the same path and the vector f depending on the increment of the scalar product of V and dx due to parallel transport with respect to (1.1) as defined above.

Again, $\because \ (g_{ij})_k + (g_{jk})_i + (g_{ki})_j = 0, \quad \therefore \ (g_{ij})_k V^i V^j V^k = 0$

$$\therefore \ (g_{ij})_k V^i V^j v^k = 0, \quad \text{if } v^k = \phi V^k, \text{ where } \phi \text{ is a scalar.}$$

Therefore, the length of the vector remains unaltered when the vector is given the parallel displacement (1.1) in its own direction.

The autoparallel described by such parallel displacement has the equations

$$\frac{d\lambda^i}{ds} + \nabla^i_{jk}\lambda^j\lambda^k = 0, \quad \text{where} \quad \lambda^i = \frac{dx^i}{ds}.$$

By (1.2), the autoparallel is a geodesic if

$$p^i \equiv -g^{it}(g_{jk})_t \lambda^j\lambda^k = 0$$

or, $$\sum_T p_T\, {}_Th^i = 0, \quad \text{where} \quad p_T = -(g_{jk})_t \lambda^j \lambda^k\, {}_Th^t.$$

Evidently, the vector p is the geodesic curvature (Levi-Civita, 1927) of the autoparallel, and the components of this vector referred to the local system (*i.e.*, its projections on ${}_Th$) are p_T. $-p_T$ measures the rate of change of the length of the unit vector λ when λ is given the parallel transport (1.1) along ${}_Th$.

Therefore an autoparallel with respect to the parallelism (1.1) is a geodesic if the length of a vector in the direction of the autoparallel remains unaltered when the vector is given the same parallel displacement in the direction of an arbitrary vector of the orthogonal ennuple.

In particular, the autoparallel described by a member ${}_Sh$ of the ennuple is a geodesic if

$$(g_{jk})_t\, {}_Sh^j\, {}_Sh^k\, {}_Th^t = 2\beta_{SST} = 0, \quad T = 1, 2, \ldots, n;$$

where the quantities β_{PQR} have been introduced in the paper referred to and called the coefficients of rotation with respect to the parallelism (1.1).

2. In the paper referred to (Sen, 1945) it was seen that

$$K_{ijkl} - R_{ijkl} = P_{ijkl} + A_{ijkl}, \quad K_{ijkl} - \bar{R}_{ijkl} = P_{ijkl} + Q_{ijkl},$$

where

$$K_{ijkl} = g_{it}K^t_{jkl}, \quad R_{ijkl} = g_{it}R^t_{jkl}, \quad \bar{R}_{ijkl} = R_{ijkl}+R_{jilk}+R_{klij}+R_{lkji}.$$

K^t_{jkl} and R^t_{jkl} are the curvature tensors formed with respect to the Levi-Civita parallelism and the parallelism (1.1) respectively, and

$$P_{ijkl} = g^{st}\{(g_{jk})_s(g_{il})_t-(g_{jl})_s(g_{ik})_t\}, \quad A_{ijkl} = (g_{jk})_{il}-(g_{jl})_{ik},$$

$$Q_{ijkl} = \sum_T[2({}^Th_j)_i({}^Th_k)_l+({}^Th_j)_l({}^Th_k)_i+({}^Th_j)_k({}^Th_i)_l].$$

We obtain below a geometrical significance of P_{ijkl}, A_{ijkl} and of Q_{ijkl} with reference to two members ${}_Ph$ and ${}_Qh$ of the orthogonal ennuple. In order to do so we consider the following vectors.

$$\{{}_Ph^i\}_t\,{}_Qh^t = ({}_Ph^i)_t\,{}_Qh^t-g^{it}(g_{jk})_t\,{}_Ph^j\,{}_Qh^k. \tag{2.1}$$

Its covariant form is

$$\{{}^Ph_i\}_t\,{}_Qh^t = ({}^Ph_i)_t\,{}_Qh^t+(g_{ij})_k\,{}_Ph^k\,{}_Qh^j. \tag{2.2}$$

Now

$$({}_Ph^i)_t\,{}_Qh^t+({}_Qh^i)_t\,{}_Ph^t = 0, \qquad \therefore \quad ({}^Ph_i)_t\,{}_Qh^t+({}^Qh_i)_t\,{}_Ph^t = -(g_{jk})_i\,{}_Ph^j\,{}_Qh^k.$$

Therefore (2.1) gives

$$\{{}_Ph^i\}_t\,{}_Qh^t+\{{}_Qh^i\}_t\,{}_Ph^t = -2g^{it}(g_{jk})_t\,{}_Ph^j\,{}_Qh^k. \tag{2.3}$$

And (2.2) gives

$$\{{}^Ph_i\}_t\,{}_Qh^t+\{{}^Qh_i\}_t\,{}_Ph^t = 2[({}^Ph_i)_t\,{}_Qh^t+({}^Qh_i)_t\,{}_Ph^t] = -2(g_{jk})_i\,{}_Ph^j\,{}_Qh^k. \tag{2.4}$$

In particular, putting $P = Q$, (2.3) and (2.4) give

$$\{{}_Ph^i\}_t\,{}_Ph^t = -g^{it}(g_{jk})_t\,{}_Ph^j\,{}_Ph^k, \tag{2.5}$$

$$\{{}^Ph_i\}_t\,{}_Ph^t = 2({}^Ph_i)_t\,{}_Ph^t = -(g_{jk})_i\,{}_Ph^j\,{}_Ph^k. \tag{2.6}$$

(2.5) and (2.6) are respectively the contravariant and covariant components of the geodesic curvature of the autoparallel described, with respect to parallelism (1.1), by ${}_Ph$.

Denote the covariant vector

$$({}^Ph_i)_t\,{}_Qh^t \text{ by } {}^P_Qh_i. \tag{2.7}$$

Obviously, the vector ${}^Ph$ is normal to ${}^Q_Ph$, and, in particular, to ${}^P_Ph$. In general,

$${}^P_Qh_i\,{}_Rh^i = -{}^P_Rh_i\,{}_Qh^i.$$

Also (2.6) gives

$${}^P_Ph_i\,{}_Qh^i = (g_{ij})_k\,{}_Ph^i\,{}_Qh^j\,{}_Ph^k. \tag{2.8}$$

Lastly,

$$\sum_P {}^P_Ph_i\,{}_Qh^i = -({}_Qh^t)_t. \tag{2.9}$$

That is, half the sum of the projections of the geodesic curvatures of all the autoparallels described, with respect to the parallelism (1.1), by the vectors of the orthogonal ennuple on one of these vectors is equal to minus the divergence of the vector with respect to the same parallelism.

Now let ϕ denote the square of the length of the vector ${}^P_Qh+{}^Q_Ph$ and ψ the scalar product of the vectors $2{}^P_Ph$ and $2{}^Q_Qh$.

Then from (2.7), (2.4) and (2.6)

$$\phi = g^{st}(g_{jk})_s(g_{il})_t\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l$$

$$\psi = g^{st}(g_{jl})_s(g_{ik})_t\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l$$

$$\therefore\quad \phi-\psi = P_{ijkl}\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l. \tag{2.10}$$

Again, by (2.8), the projection of ${}^P_Ph$ on ${}_Qh$

$$= (g_{jk})_i\, {}_Ph^i{}_Qh^j{}_Ph^k.$$

If ξ denotes the rate of change, with respect to the arc, of this projection when ${}_Ph$ and ${}_Qh$ are moved by parallel displacement (1.1) in the direction of ${}_Qh$, then

$$\xi = (g_{jk})_{il}\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l.$$

And, by (2.6) the projection of $2{}^Q_Qh$ on ${}_Ph$

$$= -(g_{jl})_i\, {}_Ph^i{}_Qh^j{}_Qh^l.$$

If χ denotes the rate of change of this projection when ${}_Ph$ and ${}_Qh$ are moved by parallel displacement (1.1) along ${}_Ph$, then

$$\chi = -(g_{jl})_{ik}\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l$$

$$\therefore\quad \xi+\chi = A_{ijkl}\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l. \tag{2.11}$$

Finally, we notice that

$$({}^Th_j)_l({}^Th_k)_i\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l = 0,$$

$$[({}^Th_j)_i({}^Th_k)_l-({}^Th_j)_k({}^Th_i)_l]\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l = 0.$$

$$\therefore\quad Q_{ijkl}\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l = 8\sum_T {}^T_Ph_s\, {}^T_Qh_t\, {}_Qh^s{}_Ph^t = -8\sum_T ({}^T_Ph_t\, {}_Qh^t)^2, \tag{2.12}$$

where ${}^T_Ph_t\, {}_Qh^t$ is the projection of ${}^T_Ph$ on ${}_Qh$.

In particular, if the autoparallels described, with respect to parallelism (1.1), by ${}_Ph$ and ${}_Qh$ are geodesics,

$${}^P_Ph = {}^Q_Qh = 0.$$

Therefore, in this case,

$$(K_{ijkl}-R_{ijkl})\, {}_Ph^i{}_Qh^j{}_Ph^k{}_Qh^l = \text{square of the length of the vector } {}^P_Qh+{}^Q_Ph.$$

It may be mentioned that the quantities ϕ, ψ, ξ, χ are respectively equal to the quantities denoted by $\sum_t F_tF_t$, $\sum_t {}_pG_{t\,q}G_t$, F_{pq}, $-{}_qG_{pp}$ in a previous paper (Sen, 1944) and obtained from a different geometrical stand-point. And a similar interpretation may be given to the quantities denoted by p_{PQRS} a_{PQRS}, q_{PQRS} in a previous paper (Sen, 1945).

3. We shall now express some fundamental invariants directly in terms of the parallelism (1.1).

We had (Sen, 1945)

$$\Delta^i_{jk} = \tfrac{1}{2}\sum_T {}_Th^i\left(\frac{\partial\, {}^Th_j}{\partial x^k}-\frac{\partial\, {}^Th_k}{\partial x^j}\right) = \sum_T {}_Th^i({}^Th_j)_k,$$

$$R^i_{jkl} = (\Delta^i_{jk})_l-(\Delta^i_{jl})_k+\Delta^i_{sl}\,\Delta^s_{jk}-\Delta^i_{sk}\,\Delta^s_{jl},$$

and

$$_1R_{jk} = \tfrac{1}{2}(R^t_{jkt} + R^t_{kjt}), \quad _1R'_{jk} = \tfrac{1}{2}(R^t_{jkt} - R^t_{kjt}), \quad _1R = g^{jk}\,{}_1R_{jk}.$$

It may be seen that

$$\Delta_j = \Delta^i_{ji} = -\sum_T g^{it}\,{}^Th_t({}^Th_i)_j = -\sum_T g^{it}\,{}^Th_i({}^Th_t)_j = -\tfrac{1}{2}g^{it}(g_{it})_j.$$

And (1.2) gives

$$T_j = T^i_{ji} = g^{it}(g_{ji})_t = -\tfrac{1}{2}g^{it}(g_{it})_j.$$

Therefore

$$-\tfrac{1}{2}g^{st}(g_{st})_j = \Delta_j = T_j = \theta_j, \text{ say.} \tag{3.1}$$

Comparing (3.1) with (2.6) it is seen that the vector θ is equal to half the sum of the geodesic curvatures of all the autoparallels described, with respect to parallelism (1.1), by the vectors of the orthogonal ennuple.

Some other forms of θ which we shall require are given below:

$$\theta_j = \tfrac{1}{2}g_{st}(g^{st})_j = g^{st}(g_{sj})_t = -\sum_T {}^Th_j({}_Th^t)_t$$

$$\theta^k = g^{jk}\theta_j = -(g^{kt})_t = -\sum_T {}_Th^k({}_Th^t)_t = -\tfrac{1}{2}g^{st}\left[\nabla^k_{st} - \begin{Bmatrix} k \\ st \end{Bmatrix}\right]$$

and

$$\theta_t\theta^t = \sum_T ({}_Th^t)_t({}_Th^s)_s.$$

Now, by (3.1),

$$(\Delta^t_{jt})_k = (\theta_j)_k, \quad \Delta^t_{sk}\Delta^s_{jt} = \sum_T ({}^Th_t)_j({}_Th^t)_k.$$

Therefore

$$-2\,{}_1R_{jk} = (\theta_j)_k + (\theta_k)_j + 2\sum_T ({}^Th_t)_j({}_Th^t)_k \tag{3.2}$$

and

$$2\,{}_1R'_{jk} = (\theta_j)_k - (\theta_k)_j. \tag{3.2a}$$

Also since

$$g^{jk}(\theta_j)_k = \theta_t\theta^t + (\theta^t)_t, \quad \sum_T g^{jk}({}^Th_t)_j({}_Th^t)_k = -\sum_T ({}_Th^t)_s({}_Th^s)_t,$$

$$\therefore \quad {}_1R = \sum_T ({}_Th^t)_s({}_Th^s)_t - \theta_t\theta^t - (\theta^t)_t. \tag{3.3a}$$

Again since

$${}_Th^t({}_Th^s)_t = 0, \quad \therefore \quad ({}_Th^t)_s({}_Th^s)_t + {}_Th^t({}_Th^s)_{ts} = 0$$

$$\therefore \quad -(\theta^t)_t = (g^{st})_{st} = \sum_T [({}_Th^t)_t({}_Th^s)_s + {}_Th^t({}_Th^s)_{st}].$$

$$\therefore \quad {}_1R = \sum_T [({}_Th^t)_s({}_Th^s)_t + {}_Th^t({}_Th^s)_{st}]. \tag{3.3b}$$

Having obtained expression for ${}_1R$ we may express the scalar curvature K directly as follows: we have

$$K = {}_1R + P + {}_1A,$$

where

$$P = g^{jk}g^{il}P_{ijkl}, \quad {}_1A = g^{jk}g^{il}A_{ijkl}.$$

It is seen that

$$P + {}_1A = 2\theta_t\theta^t - 2(\theta^t)_t + \tfrac{1}{2}g^{jk}g^{st}(g_{st})_{jk}.$$

$$\therefore \quad K = \sum_T ({}_Th^t)_s({}_Th^s)_t + \theta_t\theta^t - 3(\theta^t)_t + \tfrac{1}{2}g^{jk}g^{st}(g_{st})_{jk}. \tag{3.4a}$$

But

$$g^{jk}(\theta_j)_k = -\tfrac{1}{2}g^{jk}[(g^{st})_j(g_{st})_k + g^{st}(g_{st})_{jk}] = \theta_t\theta^t + (\theta^t)_t.$$

$$\therefore \quad K = \sum_T ({}_Th^t)_s({}_Th^s)_t - 4(\theta^t)_t - \tfrac{1}{2}g^{jk}(g^{st})_j(g_{st})_k.$$

The main differential invariants that have come out of the above conside the vector θ_j (identified with the electromagnetic potential vector in the Relativity) and the symmetric tensors $\sum_T ({}^Th_t)_j({}_Th^t)_k$ and $(g^{st})_j(g_{st})_k$.

Finally, for a contravariant vector v^i and for a skew-symmetric contravari ξ^{ij}, we had (Sen, 1945)

$$(v^i)_i = \{v^i\}_i + v^k g^{ij}(g_{ik})_j$$

$$(\xi^{ij})_{ji} = \tfrac{1}{2}\xi^{lk}g^{ij}(A_{ijkl} + B_{ijkl}).$$

Now

$$g^{ij}A_{ijkl} = -g^{ij}(A_{iklj} - A_{ilkj}) = -2\,{}_1A'_{kl} = 2\,{}_1R'_{kl} = (\theta_k)_l - (\theta_l)_k, \text{ by } (3.2a).$$

And

$$g^{ij}B_{ijkl} = 0.$$

For,

$$B_{ijkl} = g^{st}\{(g_{sk})_i(g_{jl})_t - (g_{sl})_i(g_{jk})_t\}$$

Put

$$C_{ijkl} = g^{st}\{(g_{jk})_t(g_{is})_l - (g_{jl})_t(g_{is})_k\}.$$

Then

$$B_{ijkl} = C_{ijkl} + P_{ijkl}.$$

Evidently

$$g^{ij}P_{ijkl} = 0, \quad \text{for } P_{ijkl} \text{ is skew in } i,\ j.$$

And

$$g^{ij}C_{ijkl} = -(g_{jk})_t(g^{jt})_l + (g_{jl})_t(g^{jt})_k = -(g^{jt})_l(g_{tk})_j + (g^{jt})_k(g_{tl})_j = \tfrac{1}{2}\{(g^{jt})_l(g_{jt})_k - (g^{jt})_k(g$$

Therefore, the above two divergence formulæ become

$$(v^i)_i = \{v^i\}_i + v^k\theta_k$$

$$(\xi^{ij})_{ji} = -\tfrac{1}{2}\xi^{kl}\{(\theta_k)_l - (\theta_l)_k\}.$$

Hence, if $\theta_k = 0$, for $k = 1, 2, \ldots, n$,

$$(v^i)_i = \{v^i\}_i, \quad (\xi^{ij})_{ji} = 0 = \{\xi^{ij}\}_{ji}.$$

4. In connection with the covariant vector θ_k which has been introduce with a geometrical meaning, mention may be made of the corresponding ve following characteristic equation of Weyl's geometry

$$[g_{ij}]_k + g_{ij}w_k = 0,$$

where $[\ ]_k$ denotes the covariant derivative with respect to a parallelism with connection, the covariant vector w_k not being uniquely determined by the paral

Weyl's space has no definite metric, the fundamental tensor g_{ij} being dete within a factor of multiplication called the gauge. The space is said to be calib the gauge is assigned. When the gauge is definitely fixed, the space behaves li space (Thomas, 1934).

It may not be out of place to indicate briefly, in this connection, certai

parallelism in Riemannian space which are connected with the parallelism (1.1) in different ways. In what follows θ_k is always supposed to be given by (3.1).

Let

$$dV^i + \Delta^i_{jk} V^j dx^k = 0$$

be a parallelism with Δ^i_{jk} symmetric in j, k and [ ] with a subscript denote the covariant derivative and L^t_{jkl} the curvature tensor with respect to this parallelism. Further, let ${}_1L = g^{jk} L^t_{jkt}$, and

(1) $$[g_{ij}]_k = -(2/n) g_{ij}\theta_k.$$

Then $$g^{ij}[g_{ij}]_k = n g^{ij}[g_{ik}]_j = -2\theta_k, \quad [g^{ij}]_k = (2/n) g^{ij}\theta_k,$$

$$\nabla^i_{jk} - \Delta^i_{jk} = T^i_{jk} + (1/n)\{g_{jk}\theta^i - \delta^i_j\theta_k - \delta^i_k\theta_j\},$$

$$\sum_P {}_P h^t[{}^P h_k]_t = -\sum_P {}_P h^t[{}^P h_t]_k = \theta_k, \quad \nabla^i_{ji} - \Delta^i_{ji} = 0.$$

$${}_1L - {}_1R = -\tfrac{1}{2} g^{jk}(g^{st})_j(g_{st})_k - \frac{n+2}{n}(\theta^t)_t - \frac{n-2}{n^2}\theta_t\theta^t.$$

(2) $$[g_{ij}]_k = (g_{ij})_k + g_{ik}\theta_j + g_{jk}\theta_i.$$

Then $$g^{ij}[g_{ij}]_k = 0, \quad g^{ij}[g_{ik}]_j = (n+2)\theta_k, \quad [g^{ij}]_i = -(n+2)\theta^j,$$

$$\nabla^i_{jk} - \Delta^i_{jk} = g_{jk}\theta^i,$$

$$\sum_P {}_P h^t[{}^P h_k]_t = 2\theta_k, \quad \sum_P {}_P h^t[{}^P h_t]_k = 0, \quad \nabla^i_{ji} - \Delta^i_{ji} = \theta_j,$$

$${}_1L - {}_1R = (n-1)(\theta^t)_t - (n+2)\theta_t\theta^t.$$

(3) $$[g_{ij}]_k = (g_{ij})_k + 2g_{ij}\theta_k + g_{ik}\theta_j + g_{jk}\theta_i.$$

Then $$g^{ij}[g_{ij}]_k = 2n\theta_k, \quad g^{ij}[g_{ik}]_j = (n+4)\theta_k, \quad [g^{ij}]_i = -(n+4)\theta^j,$$

$$\nabla^i_{jk} - \Delta^i_{jk} = \delta^i_j\theta_k + \delta^i_k\theta_j,$$

$$\sum_P {}_P h^t[{}^P h_k]_t = (n+2)\theta_k, \quad \sum_P {}_P h^t[{}^P h_t]_k = n\theta_k, \quad \nabla^i_{ji} - \Delta^i_{ji} = (n+1)\theta_j,$$

$${}_1L - {}_1R = \tfrac{1}{2}(n-1)\{\theta_t\theta^t + (\theta^t)_t\}.$$

It may be seen that the autoparallels with respect to the parallelism (1.1) and the parallelism given in (3) are the same if the vector θ is orthogonal to the direction of the autoparallel at every point of it. In particular, it follows from (2.9) that the congruence ${}_P h$ is an autoparallel with respect to (3) if $({}_P h^t)_t = 0$.

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RADIAL OSCILLATIONS OF A STAR

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Eddington developed the theory of sinusoidal pulsations as a first step to explain the variability of luminosity in Cepheids. The velocity-time curves and the light curves for Cepheids are not sinusoidal: in fact the time interval between the maximum velocity of approach and the maximum velocity of recession is about five times the time interval between the maximum velocity of recession and the maximum velocity of approach. Eddington's theory cannot explain this skewness of velocity-time curve. Further, the theoretical value of the period of oscillation corresponding to vanishing amplitude calculated on the "Standard Model" (Eddington, 1926) and $\gamma = \frac{5}{3}$ is about half the observed value. To explain the skewness in velocity-time curve and the discrepency in the theoretical and observed values of period, Rosseland (1943) developed the general theory of anharmonic oscillations and put the equations of motion in the Hamiltonian form:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}; \quad \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad k = 1, 2, 3, \ldots,$$

where H is the sum of kinetic energy, thermal energy and the gravitational energy of the star and q_k's are the time coefficients in the expression for displacement at a distance a from the centre given by

$$r - a = \sum_k \xi_k q_k$$

and p_k's are Hamiltonian moments

$$\frac{\partial}{\partial q_k} \int_0^M \dot{r}^2 dm = I_k \dot{q}_k.$$

Rosseland's theory does not provide with the method of determining ξ_k and there is no justification in assuming it for anharmonic oscillation to be the same as for sinusoidal oscillation, though as an approximation we may take it to be same in both the cases. It is to be noted that in the theories of sinusoidal and anharmonic oscillations, oscillation has been assumed to have the character of standing wave throughout the star.

In the present note, expressions for period and skewness, for the fundamental mode of oscillation, have been obtained correct to square of amplitude at the surface of the star and then the results have been applied to three models introduced by Sterne (1937) assuming in each case ξ_k to be same as for oscillation with vanishing amplitude. Further, an attempt has been made to determine the displacement function ξ_k by considering the equation of motion derived by the "displacement

method" introduced by Eddington in the case of sinusoidal oscillation. The analysis suggests that probably there is no model with varying density in which the oscillation has the character of standing wave throughout the entire model and that in the envelope of such a model its character, as a necessity, changes to that of progressive wave.

1. Let r, ρ, p, T and g denote the radius vector, density, pressure, temperature and gravity of an element in the Lagrangian sense and a, ρ_0, p_0, T_0 and g_0 denote these quantities at a given intial time t_0. In the sequel, we shall identify t_0 with the instant when the velocity of oscillation is maximum, it being directed outwards.

The matter in the spherical shell r to $r+dr$ occupies in the undisturbed state the shell a to $a+da$; hence, the equation of continuity is

$$\rho r^2 dr = \rho_0 a^2 da,$$

which gives

$$\rho/\rho_0 = (a^2/r^2)(1/r'), \tag{1}$$

where dash denotes differentiation with respect to a.

For adiabatic changes, the pressure and density of a particular element of matter are connected by the relation

$$p/p_0 = (\rho/\rho_0)^\gamma,$$

where γ is the ratio of specific heats. Using (1), it reduces to

$$p/p_0 = (a^2/r^2)^\gamma (1/r')^\gamma. \tag{2}$$

The kinetic energy $W_1(t)$ of radial oscillation is

$$W_1 = \tfrac{1}{2}\int_0^M \dot{r}^2 dm = 2\pi \int_0^R \dot{r}^2 a^2 \rho_0 da, \tag{3}$$

where M is the mass of the star and R its radius at t_0.

The work $W_2(t)$ done against gravitation, measured from the state at time t_0, is

$$W_2 = -\int_0^M g_0(a^2/r)dm + \int_0^M g_0 a\,dm = 4\pi \int_0^R a^3 \rho_0 g_0 (1-a/r) da. \tag{4}$$

The increase in thermal energy $W_3(t)$ also measured from the state at t_0 is

$$W_3 = \int_0^M c_v(T-T_0)dm,$$

where c_v is the constant volume specific heat per unit mass. Substituting the adiabatic relation (2) and the perfect gas equation

$$p = c_v(\gamma-1)\rho T,$$

we get

$$W_3 = \frac{4\pi}{\gamma-1}\int_0^R p_0 a^2 \left\{\left(\frac{a^2}{r^2}\frac{1}{r'}\right)^{\gamma-1} - 1\right\} da. \tag{5}$$

Adding (3), (4) and (5) we have for the total energy H, which does not vary with time,

$$H = 4\pi \int_0^R a^2\rho_0 \left[\tfrac{1}{2}\dot{r}^2 + a g_0\left(1-\frac{a}{r}\right) + \frac{1}{\gamma-1}\frac{p_0}{\rho_0}\left\{\left(\frac{a^2}{r^2}\frac{1}{r'}\right)^{\gamma-1} - 1\right\}\right] da. \tag{6}$$

Differentiating (6) with respect to t, we get

$$0 = \int_0^R a^2\rho_0\left[\dot{r}\ddot{r} + \frac{a^2}{r^2}g_0\dot{r} - \frac{p_0}{\rho_0}\left\{\frac{2a^{2\gamma-2}\dot{r}}{r^{2\gamma-1}(r')^{\gamma-1}} + (a/r)^{2\gamma-2}(1/r')^{\gamma}\frac{dr'}{dt}\right\}\right]da$$

$$= \int_0^R a^2\rho_0\dot{r}\left[\ddot{r} + g_0\{(a/r)^2 - (a/r)^{2\gamma-2}(1/r')^{\gamma}\} + \frac{p_0\gamma}{\rho_0}(a/r)^{2\gamma-2}(1/r')^{\gamma}\left\{\frac{2}{a} - \frac{2r'}{r} - \frac{r''}{r'}\right\}\right]da, \quad (7)$$

since

$$\int_0^R p_0\frac{a^{2\gamma}}{r^{2\gamma-2}}(1/r')^{\gamma}\frac{dr'}{dt}da = \int_0^R p_0\frac{a^{2\gamma}}{r^{2\gamma-2}}(1/r')^{\gamma}\frac{d\dot{r}}{da}da$$

$$= -\int_0^R \dot{r}\frac{a^{2\gamma}}{r^{2\gamma-2}}(1/r')^{\gamma}\left\{-g_0\rho_0 + p_0\left(\frac{2\gamma}{a} - \frac{2\gamma-2}{r}r' - \frac{\gamma}{r'}r''\right)\right\}da,$$

as the integrated part vanishes at both the limits and

$$\frac{dp_0}{da} = -g_0\rho_0. \quad (8)$$

The equation (7) is satisfied when

$$\ddot{r} + g_0\{(a/r)^2 - (a/r)^{2\gamma-2}(1/r')^{\gamma}\} + \frac{p_0\gamma}{\rho_0}(a/r)^{2\gamma-2}(1/r')^{\gamma}\left\{\frac{2}{a} - \frac{2r'}{r} - \frac{r''}{r'}\right\} = 0. \quad (9)$$

(9) is the equation which governs the adiabatic radial oscillations of a gaseous star and which can be directly obtained following the method introduced by Eddington (1926) for the case of oscillation with vanishing amplitude and which consists in eliminating p and ρ in the equation of motion

$$\frac{\partial p}{\partial r} + \rho\ddot{r} + \rho\frac{GM_r}{r^2} = 0,$$

with the help of (1), (2) and (8), where M_r denotes the mass of the star within a sphere of radius r. We shall refer (6) as the "energy equation".

2. If the oscillation has the character of standing wave, the displacement at any point a for the fundamental mode can be written in the form

$$r - a = r\eta(a)q(t), \quad (10)$$

where q is the function of time only and η the function of a only, having the form

$$\eta = \eta_B f(a/R), \quad (11)$$

where η_B is the value of η at the boundary, so that $f = 1$ at $a = R$. Substituting (10) in (6) we get

$$H = 2\pi\dot{q}^2\int_0^R \rho_0\eta^2a^4da + 4\pi\int_0^R \rho_0 g_0 a^3\frac{\eta q}{1+\eta q}da$$

$$+\frac{4\pi}{\gamma-1}\int_0^R p_0a^2\left\{\frac{1}{(1+\eta q)^{2\gamma-2}(1+\eta q+a\eta' q)^{\gamma-1}} - 1\right\}da \quad (12)$$

$$= A\dot{q}^2 + Bq^2 - Cq^3 + Dq^4, \quad (13)$$

neglecting powers of q higher than the fourth, where

$$\left.\begin{aligned}
A &= 2\pi\int_0^R \eta^2\rho_0 a^3 da,\\
B &= 2\pi(3\gamma-4)\int_0^R \eta^2\rho_0 g_0 a^3 da + 2\pi\gamma\int_0^R (\eta')^2 p_0 a^4 da,\\
C &= \tfrac{2}{3}\pi(3\gamma-4)(3\gamma+1)\int_0^R \eta^3\rho_0 g_0 a^3 da + \tfrac{2}{3}\pi\int_0^R p_0 a^4\{3\gamma(3\gamma-1)\eta(\eta')^2+\gamma(\gamma+1)a(\eta')^3\}da,\\
\text{and}\quad D &= \tfrac{1}{2}\pi(3\gamma-4)(3\gamma^2+\gamma+2)\int_0^R \eta^4\rho_0 g_0 a^3 da + \tfrac{1}{6}\pi\gamma\int_0^R p_0 a^2\{18\gamma(3\gamma-1)\eta^2 a^2(\eta')^2\\
&\qquad\qquad + 12\gamma(\gamma+1)\eta a^3(\eta')^3+(\gamma+1)(\gamma+2)a^4(\eta')^4\}da,
\end{aligned}\right\} \quad (14)$$

since the coefficient of q namely

$$4\pi\left[\int_0^R p_0 g_0\eta a^3 da - \int_0^R p_0\frac{d}{da}(\eta a^3)da\right] = 4\pi\left[\int_0^R \rho_0 g_0\eta a^3 da + \int_0^R \eta a^3\frac{dp_0}{da}da\right] = 0, \text{ by (8).}$$

It will be seen from (11) that $C/B = \alpha\eta_B$ and $D/B = \beta\eta_B^2$, where α and β are independent of η_B. Let now η be the actual maximum positive extension in the radial direction at a during any cycle, then $\dot q = 0$ when $q = 1$ and η_B will be the amplitude* at the surface. For the fundamental mode of oscillation η is positive and does not decrease with increasing a so that A, B, C, and D are positive for $\gamma > \frac{4}{3}$. Using the condition $\dot q = 0$ when $q = 1$, (13) may be written as

$$\dot q^2 = \sigma_1^2(1-q)\left[\left(1-\frac{C}{B}+\frac{D}{B}\right)+\left(1-\frac{C}{B}+\frac{D}{B}\right)q-\left(\frac{C}{B}-\frac{D}{B}\right)q^2+\frac{D}{B}q^3\right],$$

where $\sigma_1^2 = B/A$. When $q = 0$, $\dot q$ is real and hence $1-(C/B)+(D/B)$ is positive. The expression within the square brackets has opposite signs when $q = 0$ and $q = -1$, therefore it has a zero $q = -e$, $0 < e < 1$. We can now write down the above expression for $\dot q^2$ in the form

$$\dot q^2 = \sigma_1^2(1-q)(e+q)\left[\left(1-\frac{C}{B}+\frac{D}{B}\right)-\left(\frac{C}{B}-\frac{D}{B}\right)(q-e)+(D/B)(q^2-eq+e^2)\right].$$

The value of e, correct to the square of amplitude at the surface, is

$$e = 1-\alpha\eta_B+\alpha^2\eta_B^2$$

so that to this approximation

$$\dot q^2 = \sigma_1^2(1-q)(e+q)\left[1-(\alpha^2-\beta)\eta_B^2-\alpha\eta_B q+\beta\eta_B^2 q^2\right]. \quad (15)$$

* The amplitude as defined in this paper is dimensionless, being the ratio of displacement at the surface to the radius.

Separating the variables q and t and setting $q = \sin^2\theta - e\cos^2\theta$, we have

$$dt = \frac{2}{\sigma_1} f(\theta)\,d\theta, \tag{16}$$

where

$$f(\theta) = 1 + \tfrac{1}{2}(\alpha^2-\beta)\eta_B^2 - \tfrac{1}{2}\alpha\eta_B \cos 2\theta + \tfrac{1}{2}\alpha^2\eta_B^2 \cos^2\theta - \tfrac{1}{2}(\beta - \tfrac{3}{4}\alpha^2)\eta_B^2 \cos^2 2\theta.$$

The period of oscillation P, and t_1 and t_2, the parts of the period for which the radius of the star is greater and less than R respectively, are immediately obtained * from (16):

$$P = \frac{4}{\sigma_1}\int_0^{\frac{\pi}{2}} f(\theta)\,d\theta = \frac{2\pi}{\sigma_1}\left[1 + \tfrac{3}{4}(\tfrac{5}{4}\alpha^2 - \beta)\eta_B^2\right] = \frac{2\pi}{\sigma_1}\left[1 + \frac{3}{4}\left(\frac{5}{4}\frac{C^2}{B^2} - \frac{D}{B}\right)\right], \tag{17}$$

$$t_2 = \frac{4}{\sigma_1}\int_0^{\theta_0} f(\theta)\,d\theta \quad \text{and} \quad t_1 = \frac{4}{\sigma_1}\int_{\theta_0}^{\frac{\pi}{2}} f(\theta)\,d\theta,$$

where

$$\sin\theta_0 = \sqrt{\left(\frac{e}{1+e}\right)} = \frac{1}{\sqrt{2}}\left(1 - \tfrac{1}{4}\alpha\eta_B + \tfrac{3}{32}\alpha^2\eta_B^2\right) = \frac{1}{\sqrt{2}}\left\{1 - \tfrac{1}{4}(C/B) + \tfrac{3}{32}(C^2/B^2)\right\},$$

correct to our approximation.

The skewness δ is given by

$$\delta = \frac{t_1}{t_2} = \int_{\theta_0}^{\frac{\pi}{2}} f(\theta)\,d\theta \Bigg/ \int_0^{\theta_0} f(\theta)\,d\theta = 1 + \frac{4}{\pi}\frac{C}{B} + \left(\frac{8}{\pi^2} - \frac{7}{4\pi}\right)\frac{C^2}{B^2}. \tag{18}$$

We shall now calculate P and δ for some special models:

Model 1. '$\rho_0 =$ constant': For this model, the values of P and δ have been already calculated for all values of η_B and for $\gamma = \frac{5}{3}, \frac{3}{2}, \frac{13}{9}$ (Bhatnagar, 1946).

Here

$$p_0 = \tfrac{1}{2}\rho_0\frac{GM}{R}\left(1 - \frac{a^2}{R^2}\right), \quad g_0 = \frac{GM}{R^3}a \quad \text{and} \quad \eta = \eta_B;$$

hence

$$\sigma_1^2 = \frac{MG}{R^3}(3\gamma-4), \quad C/B = \tfrac{1}{3}(3\gamma+1)\eta_B, \quad D/B = \tfrac{1}{4}(3\gamma^2+\gamma+2)\eta_B^2,$$

$$\frac{P}{P_0} = \frac{1}{\sqrt{(3\gamma-4)}}\left[1 + \tfrac{1}{48}(18\gamma^2+21\gamma-13)\eta_B^2\right], \quad \delta = 1 + \frac{4}{3\pi}(3\gamma+1)\eta_B + \frac{1}{9}\left(\frac{8}{\pi^2} - \frac{7}{4\pi}\right)(3\gamma+1)^2\eta_B^2,$$

where

$$P_0 = 2\pi\sqrt{(R^3/MG)}.$$

Model 2. '$\rho_0 \propto 1/a^2$': For this model

$$\rho_0 = \frac{M}{4\pi R}\frac{1}{a^2}, \quad p_0 = \frac{M^2G}{8\pi R^2}\left(\frac{1}{a^2} - \frac{1}{R^2}\right), \quad g_0 = \frac{MG}{R}\frac{1}{a},$$

$$\eta = \eta_B(a/R)^l, \quad l = \tfrac{1}{2}\left[-1 + \sqrt{(1+8\alpha)}\right], \quad \alpha = 3 - \frac{4}{\gamma};$$

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* It will be seen from the expression for P obtained here that to get P correct to square of the amplitude, we should retain in (13) powers of q up to the fourth and not upto the third.

hence

$$\sigma_1^2 = \frac{MG}{R^3}\left[(3\gamma-4)\frac{2l+3}{2l+1}+\gamma\frac{l^2}{2l+1}\right],$$

$$\frac{C}{B} = \tfrac{1}{3}\eta_B\frac{(2l+1)(2l+3)}{(3l+1)(l+1)}\cdot\frac{(3\gamma-4)(3\gamma+1)(3l+3)+3\gamma(3\gamma-1)l^2+\gamma(\gamma+1)l^3}{(3\gamma-4)(2l+3)+\gamma l^2},$$

and

$$\frac{D}{B} = \frac{\eta_B^2}{4}\cdot\frac{(2l+1)(2l+3)}{(4l+1)(4l+3)}\cdot\frac{(3\gamma-4)(3\gamma^2+\gamma+2)(4l+3)+\frac{1}{3}\gamma l^2\{18\gamma(3\gamma-1)+12\gamma(\gamma+1)l+(\gamma+1)(\gamma+2)l^2\}}{(3\gamma-4)(2l+3)+\gamma l^2}.$$

Model 3. We assume here that the mass consists of a particle of mass $M_c = M-4\pi R\phi$ at the centre and a continuous density distribution $\rho_0 = \phi/a^2$ at all points in the boundary, where ϕ is constant, which we ultimately allow to be zero.

Here

$$p_0 = \frac{M_c G\phi}{3}\left(\frac{1}{a^3}-\frac{1}{R^3}\right)+2\pi G\phi^2\left(\frac{1}{a^2}-\frac{1}{R^2}\right); \quad g_0 = \frac{M_c G}{a^2}+\frac{4\pi\phi G}{a}$$

and

$$\eta = \eta_B(a/R)^l, \quad l = \sqrt{(3\alpha)}.$$

As $\phi\to 0$, $\rho_0\to 0$, $p_0\to 0$ except at the centre and $M_c\to M$, and $g_0\to MG/a^2$. For this model,

$$\sigma_1^2 \to \frac{MG}{R^3}(3\gamma-4)(1+3/l), \quad C/B\to \tfrac{2}{3}\eta_B\frac{(2l+3)}{(l+1)(l+3)}\{(2\gamma+1)l+(6\gamma-1)\},$$

and

$$D/B\to \tfrac{1}{16}\eta_B^2\frac{2l+3}{l+3}\left\{(3\gamma^2+\gamma+2)+\frac{18\gamma(3\gamma-1)+12\gamma(\gamma+1)l+(\gamma+1)(\gamma+2)l^2}{4l+3}\right\}.$$

In the following table, the values of P/P_0 and δ in terms of η_B have been given for $\gamma = \frac{5}{3}$ and $\frac{3}{2}$:

$\gamma = \frac{5}{3}$

Model	1	2	3
P/P_0	$1+\frac{3}{2}\eta_B^2$	$0{\cdot}6783(1+1{\cdot}043\eta_B^2)$	$0{\cdot}5559(1+0{\cdot}905\eta_B^2)$
δ	$1+2{\cdot}5465\eta_B+1{\cdot}014\eta_B^2$	$1+2{\cdot}2797\eta_B+0{\cdot}813\eta_B^2$	$1+2{\cdot}3431\eta_B+0{\cdot}859\eta_B^2$

$\gamma = \frac{3}{2}$

P/P_0	$\sqrt{2}(1+\frac{59}{48}\eta_B^2)$	$0{\cdot}9182(1+0{\cdot}915\eta_B^2)$	$0{\cdot}7071(1+0{\cdot}763\eta_B^2)$
δ	$1+2{\cdot}3843\eta_B+0{\cdot}852\eta_B^2$	$1+2{\cdot}1162\eta_B+0{\cdot}700\eta_B^2$	$1+2{\cdot}1221\eta_B+0{\cdot}704\eta_B^2$

In the following table, the values of $\frac{15}{16}(C^2/B^2\eta_B^2)$ and $\frac{3}{4}(D/B\eta_B^2)$ have been given to show the importance of retaining the q^4 term in (13) for these models:

γ	$\frac{5}{3}$		$\frac{3}{2}$	
Model	$\frac{15}{16}(C^2/B^2\eta_B^2)$	$\frac{3}{4}(D/B\eta_B^2)$	$\frac{15}{16}(C^2/B^2\eta_B^2)$	$\frac{3}{4}(D/B\eta_B^2)$
1	3·750	2·250	3·151	1·922
2	3·005	1·962	2·590	1·675
3	3·175	2·270	2·604	1·842

To know upto what value of η_B approximately, the above formulæ for P/P_0 are useful, we consider the *Model* 1 for which the period is known for all values of η_B and for $\gamma = \frac{5}{3}$ (Bhatnagar and Kothari, 1944):

$$\frac{P}{P_0} = \frac{(1+\eta_B)^3}{(1+2\eta_B)^{3/2}} = 1+\tfrac{3}{2}\eta_B^2 - 3\eta_B^3,$$

correct to the third power of amplitude at the surface.

In order that we may be justified in neglecting the term in η_B^3 in the expression for P/P_0,

$$\eta_B \ll \tfrac{1}{2}.$$

In particular, if $3\eta_B^3 = \frac{1}{10}(\frac{3}{2}\eta_B^2)$ *i.e.*, $\eta_B = \frac{1}{20}$, the formula $P/P_0 = 1+\frac{3}{2}\eta_B^2$ gives the value of P/P_0, correct to three places of decimal. For $\eta_B = \frac{1}{10}$, the complete formula gives $P/P_0 = 1·0125$ and the approximate formula correct to η_B^2 gives $P/P_0 = 1·0150$ and thus roughly speaking only first two places of decimal are reliable in the value given by approximate formula. For the other models also, the limits for η_B cannot be very much different from those found above for the homogeneous model.

3. We shall now consider the equation (9). Substituting the value of displacement given by (10) and neglecting the terms containing powers of q higher than the second, equation (9) reduces to

$$\ddot{q} = -\sigma_2^2 q + Eq^2, \tag{19}$$

where

$$\sigma_2^2 = (1/a\eta)[g_0\{(3\gamma-4)\eta+\gamma a\eta'\} - (p_0\gamma/\rho_0)(4\eta'+a\eta'')], \tag{20}$$

and

$$E = (1/a\eta)[g_0\{\tfrac{1}{2}(3\gamma-4)(3\gamma+1)\eta^2+\gamma(3\gamma-1)a\eta\eta'+\tfrac{1}{2}\gamma(\gamma+1)a^2(\eta')^2\}$$
$$-(p_0\gamma/\rho_0)\{4(3\gamma-1)\eta\eta'+2(2\gamma+1)a(\eta')^2+(3\gamma-1)a\eta\eta''+(\gamma+1)a^2\eta'\eta''\}]. \tag{21}$$

If the oscillation has the character of standing wave, (20) and (21), regarded as differential equations governing η, should be satisfied by the same value of η, so that σ_2^2 and E in (19) are constants and then q, which we have taken to be function of t only on the assumption of standing wave would be determined by (19).

The solutions of (20) for the three models considered in §2, which are regular at the centre, are given by

Model 1: $\eta = \eta_B$

Model 2: $\eta = \eta_B (a/R)^l, \; l = \frac{1}{2}[-1+\sqrt{(1+8\alpha)}]$

Model 3: $\eta = \eta_B (a/R)^l, \; l = \sqrt{(3\alpha)}$.

For the Model 1, when $\eta = \eta_B$ is substituted in (21), E is determined:

$$E = \frac{MG}{2R^3}(3\gamma-4)(3\gamma+1)\eta_B = \text{constant}$$

and hence the assumption of standing wave is justified. Not only this, if in (19), higher powers of q be retained and in the coefficients of $q^3, q^4, \ldots,$ we put $\eta = \eta_B$, then we find that these coefficients come out to be independent of a. Hence for this model $\eta = \eta_B$ is the amplitude function in general.

For the Model 2, when $\eta = \eta_B (a/R)^l$ is substituted in (21), we get

$$E = \frac{MG}{R^3}\eta_B (a/R)^{l-2}[\{\tfrac{1}{2}(3\gamma-4)(3\gamma+1)l + \tfrac{1}{2}\gamma(\gamma+1)l^2\} - \tfrac{1}{2}\gamma(1-a^2/R^2)\{4(3\gamma-4)l$$
$$+2(2\gamma+1)l^2+(3\gamma-1)l(l-1)+(\gamma+1)l^2(l-1)\}],$$

and thus E is a function of a. Integrating (19) we shall have period as a function of a, meaning thereby that the period is different for different spherical layers of the model; in §2, we have obtained, with the help of energy equation, an average estimate of the dependence of period and skewness on amplitude. Such an oscillation cannot be stable and perhaps the internal friction will tend to reduce the amplitude, the period ultimately tending to its value corresponding to vanishing amplitude. The other alternative is to discard the assumption of standing wave character of oscillation throughout the model. In the interior of the model where the amplitude is small, oscillation may have the character of standing wave but in the envelope, the character changes to that of progressive wave. Such an oscillation has the period in which the interior of the model, where standing wave prevails, is oscillating and the progressive wave character of oscillation in the envelope puts the light-variation and velocity-variation at the surface in phase (Schwarzchild, 1938), explaining to a great extent the quarter period retardation of phase in light-variation—the proposed oscillation gives maximum light at the instant of maximum velocity of approach, where as the assumption of standing wave throughout the model gives maximum light at the instant of greatest compression.

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ON CERTAIN INTEGRALS INVOLVING LEGENDRE AND BESSEL FUNCTIONS

By

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1. In a recent paper B. N. Bose (1944) has obtained certain new connections between Legendre and Bessel functions and has evaluated certain infinite Integrals involving Bessel functions. The object of this paper is to extend, to generalise some of his results and to obtain some new similar integrals.

2. Consider the integral,

$$I_1 \equiv \int_0^1 P_n(1-2y^2)J_m(yz)y^{\nu-1}dy, \quad m+\nu > 0.$$

Writing the equivalent infinite series for $J_m(yz)$ and integrating term by term with the help of the integral (Cooke, 1924)

$$\int_0^1 P_n(1-2y^2)y^{2m+1}\,dy = \frac{(-)^n\{\Gamma(m+1)\}^2}{2\Gamma(m-n+1)\Gamma(m+n+2)},$$

$m > -1$ and n zero or $+ve$ integer, we get,

$$I_1 \equiv \frac{(-)^n z^m}{2^{m+1}} \sum_{r=0}^{\infty} \frac{(-)^r\{\Gamma(r+\frac{1}{2}m+\frac{1}{2}\nu)\}^2(z^2/4)^r}{r!\Gamma(r+m+1)\Gamma(r+\frac{1}{2}m+\frac{1}{2}\nu-n)\Gamma(r+\frac{1}{2}m+\frac{1}{2}\nu+n+1)}. \tag{2.1}$$

The term by term integration is justified on account of the uniform convergence of the Bessel functions in the interval of integration.

If now we reduce the series on the right of (2.1), we get that

$$\int_0^1 P_n(1-2y^2)J_m(yz)y^{\nu-1}\,dy = \frac{(-)^n z^m\{\Gamma(\frac{1}{2}m+\frac{1}{2}\nu)\}^2}{2^{m+1}\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-n)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+n+1)}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}m+\frac{1}{2}\nu,\ \frac{1}{2}m+\frac{1}{2}\nu\,; \\ m+1,\ \frac{1}{2}m+\frac{1}{2}\nu-n,\ \frac{1}{2}m+\frac{1}{2}\nu+n+1\,;\end{matrix}\ -z^2/4\right], \quad m+\nu > 0.$$

If we put $\nu = m+2$, it is easy to see that our formula gives the known integral (Bose, 1944)

$$\int_0^1 P_n(1-2y^2)J_m(yz)y^{m+1}dy = \frac{(-)^n z^m\Gamma(m+1)}{2^{m+1}\Gamma(m-n+1)\Gamma(m+n+2)}$$

$$\times {}_1F_2[m+1,\ m-n+1,\ m+n+2\,;\ -z^2/4], \quad m \geq -1.$$

3. Multiplying either side of the result (Mital, 1941)

$$\int_0^1 P_n(x)J_m(2px)x^{\nu-\frac{1}{2}}\,dx = \frac{p^m\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4};\\ m+1,\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4},\ \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4};\end{matrix}\ -p^2\right],\quad m+\nu+\tfrac{1}{2}>0.$$

by h^n and varying n from 0 to ∞, we have

$$I_2 \equiv \sum_{n=0}^{\infty} h^n \int_0^1 P_n(x)J_m(2px)x^{\nu-\frac{1}{2}}\,dx = \sum_{n=0}^{\infty}\frac{h^np^m\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4};\\ m+1,\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4},\ \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4};\end{matrix}\ -p^2\right].$$

If we use well known generating function of Legendres' Polynomial

$$\frac{1}{[1-2hx+h^2]^{1/2}} = \sum_{n=0}^{\infty} h^nP_n(x)$$

we get, after interchanging the order of integration and summation, that

$$\int_0^1 \frac{J_m(2px)x^{\nu-\frac{1}{2}}\,dx}{[1-2hx+h^2]^{1/2}} = \sum_{n=0}^{\infty}\frac{h^np^m\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4};\\ m+1,\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4},\ \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4};\end{matrix}\ -p^2\right],\quad m+\nu+\tfrac{1}{2}>0.$$

For $m=\frac{1}{2}$ and $-\frac{1}{2}$, since the Bessel function reduces to circular functions, we get as particular cases of I_2, the results :—

(1) $$\int_0^1 \frac{x^{\nu-1}\sin(2px)}{[1-2hx+h^2]^{1/2}}\,dx = \frac{\sqrt{\pi}.p\Gamma(\nu+1)}{2^\nu}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2})\Gamma(\frac{1}{2}\nu-\frac{1}{2}n+1)}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}\nu+1;\\ \frac{3}{2},\ \frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2},\ \frac{1}{2}\nu-\frac{1}{2}n+1;\end{matrix}\ -p^2\right],\quad \nu>-1,$$

(2) $$\int_0^1 \frac{x^{\nu-1}\cos(2px)}{[1-2hx+h^2]^{1/2}}\,dx = \frac{\sqrt{\pi}.\Gamma(\nu)}{2^\nu}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}n+1)\Gamma(\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}\nu,\ \frac{1}{2}\nu+\frac{1}{2};\\ \frac{1}{2},\ \frac{1}{2}\nu+\frac{1}{2}n+1,\ \frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2};\end{matrix}\ -p^2\right],\quad \nu>0.$$

4. If we apply the method of §3 to the integral (Mital, 1941)

$$\int_0^1 P_n(x)J_l(2px)J_m(2px)x^{\nu-\frac{1}{2}}\,dx$$
$$= \frac{p^{l+m}\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(l+1)\Gamma(m+1)\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4})}$$
$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4},\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}l+\frac{1}{2}m+1;\\ l+1,\ m+1,\ l+m+1,\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4},\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4};\end{matrix}\ -4p^2\right],$$
$$l+m+\nu+\tfrac{1}{2}>0,$$

we get, when $l+m+\nu+\frac{1}{2}\geq 0$,

$$I_3 \equiv \int_0^1 \frac{J_l(2px)J_m(2px)}{[1-2hx+h^2]^{1/2}}x^{\nu-\frac{1}{2}}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{h^n p^{l+m}\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(l+1)\Gamma(m+1)\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4})}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}l+\frac{1}{2}m+1,\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4}\,; \\ l+1,\ m+1,\ l+m+1,\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{5}{4},\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{3}{4}\,;\end{matrix}\ -4p^2\right].$$

It is easy to deduce the following formulae as the particular cases of the above results :—

(1) $$\int_0^1 \frac{J_m(2px)\sin(2px)}{[1-2hx+h^2]^{1/2}}x^{\nu-1}\,dx$$

$$=\frac{\sqrt{\pi}.p^{m+1}\Gamma(m+\nu+1)}{2^{\nu+m}\Gamma(m+1)}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+1)}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}m+\frac{3}{4},\ \frac{1}{2}m+\frac{5}{4},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}m+\frac{1}{2}\nu+1\,; \\ \frac{3}{2},\ m+1,\ m+\frac{3}{2},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2},\ \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+1)\,;\end{matrix}\ -4p^2\right],$$

(2) $$\int_0^1 \frac{J_m(2px)\cos(2px)x^{\nu-1}}{[1-2hx+h^2]^{1/2}}\,dx = \frac{\sqrt{\pi}.p^m\Gamma(m+\nu)}{2^{\nu+m}\Gamma(m+1)}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2})}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}m+\frac{1}{4},\ \frac{1}{2}m+\frac{3}{4},\ \frac{1}{2}m+\frac{1}{2}\nu,\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}\,; \\ \frac{1}{2},\ m+1,\ m+\frac{1}{2},\ \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+1,\ \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2}\,;\end{matrix}\ -4p^2\right],$$

(3) $$\int_0^1 \frac{\cos(4px)-1}{[1-2hx+h^2]^{1/2}}x^{\nu-\frac{3}{2}}\,dx = \frac{2\sqrt{\pi}.p^2\Gamma(\nu+\frac{3}{2})}{2^{\nu-\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}n+\frac{7}{4})\Gamma(\frac{1}{2}\nu-\frac{1}{2}n+\frac{5}{4})}$$

$$\times {}_3F_4\left[\begin{matrix}1,\ \frac{1}{2}\nu+\frac{3}{4},\ \frac{1}{2}\nu+\frac{5}{4}\,; \\ \frac{3}{2},\ 2,\ \frac{1}{2}\nu+\frac{1}{2}n+\frac{7}{4},\ \frac{1}{2}\nu-\frac{1}{2}n+\frac{5}{4}\,;\end{matrix}\ -4p^2\right],$$

(4) $$\int_0^1 \frac{\cos(4px)+1}{[1-2hx+h^2]^{3/2}}x^{\nu-\frac{3}{2}}\,dx = \frac{2\sqrt{\pi}.\Gamma(\nu-\frac{1}{2})}{2^{\nu-\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{h^n}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{4})\Gamma(\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{4})}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}\nu-\frac{1}{4},\ \frac{1}{2}\nu+\frac{1}{4}\,; \\ \frac{1}{2},\ \frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{4},\ \frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{4}\,;\end{matrix}\ -4p^2\right],$$

5. From the result (15) of B. N. Bose (1944), we have

$$\int_0^1 \frac{\sin(2y/p)\,dy}{[(1-h)^2+4hy^2]^{1/2}} = (\pi/2)\sum_{n=0}^{\infty} h^n J_{n+\frac{1}{2}}(1/p). \qquad (5.1)$$

Now, let us suppose that $x \doteqdot 1/p$. Since bei $(2\sqrt{x}) \doteqdot \sin(1/p)$, the original of the left hand side of (5.1) is

$$\int_0^1 \frac{\text{bei}.\{2\sqrt{(2xy)}\}\,dy}{[(1-h)^2+4hy^2]^{1/2}}.$$

Again
$$J^2_{n+\frac{1}{2}}(1/p) = \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(2m+2n+2)(1/2p)^{2m+2n+1}}{m!\Gamma(m+2n+2)\{\Gamma(m+n+\frac{3}{2})\}^2}$$

$$\div \sum_{m=0}^{\infty} \frac{(-)^m (x/2)^{2m+2n+1}}{m!\Gamma(m+2n+2)\{\Gamma(m+n+\frac{3}{2})\}^2} = \frac{(x/2)^{2n+1}}{\Gamma(2n+2)\{\Gamma(n+\frac{3}{2})\}^2} \times {}_0F_3[2n+2, n+\tfrac{3}{2}, n+\tfrac{3}{2}; -x^2/4].$$

Now by Lerch's Theorem, the originals of both sides of (5.1) must be the same. Hence

$$I_5 \equiv \int_0^1 \frac{\text{bei}\ \{2\sqrt{(2xy)}\}dy}{[(1-h)^2+4hy^2]^{1/2}} = (\pi/2) \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1} h^n}{\Gamma(2n+2)\{\Gamma(n+\frac{3}{2})\}^2} \times {}_0F_3[2n+2,\ n+\tfrac{3}{2},\ n+\tfrac{3}{2};\ -x^2/4].$$

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INTEGRALS INVOLVING LEGENDRE AND BESSEL FUNCTIONS

By

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1. The object of the paper is to evaluate integrals of the type

$$I_m \equiv \int_0^1 P_{n+\frac{1}{2}}(x) \prod_{\xi=q_1}^{q_m} J_\xi(2px)x^{\nu-\frac{1}{2}}\,dx, \quad t_s+\nu+1>0,$$

and

$$I'_q \equiv \int_0^1 P_n(1-2y^2) \prod_{\xi=p_1}^{p_q} J_\xi(yz)y^{\nu-1}\,dy, \quad \nu+p_1+p_2+\cdots p_q>0.$$

It is interesting to note that for the second integral

$$I'_q \equiv (z/2)^{p_q} \sum_{s_{q-2}=0}^{\infty} \frac{(-)^{s_{q-2}}(z/2)^{2s_{q-2}}}{\Gamma(s_{q-2}+1)\Gamma(s_{q-2}+p_q+1)} I'_{q-1}, \tag{1.1}$$

when for ν in I'_{q-1}, we write $\nu+2s_{q-2}+p_q$.

2. To evaluate the first integral, consider m to be odd and equal to $2\lambda+1$, $\lambda=0, 1, \ldots, \nu$. We shall now adopt the following notations:

$$R(r_\lambda) = \Gamma(r_\lambda+1)\Gamma(r_{\lambda-1}+1)\ldots\Gamma(r_1+1),$$

$$Q(q_{2\lambda+1}, q_{2\lambda}, r_\lambda, 1) = \Gamma(q_{2\lambda+1}+q_{2\lambda}+r_\lambda+1)\Gamma(q_{2\lambda-1}+q_{2\lambda-2}+r_{\lambda-1}+1)\ldots\Gamma(q_3+q_2+r_1+1),$$

$$Q(q_{2\lambda+1}, r_\lambda, 1;\, q_{2\lambda}, r_\lambda, 1) = \Gamma(q_{2\lambda+1}+r_\lambda+1)\Gamma(q_{2\lambda}+r_\lambda+1)\ldots\Gamma(q_3+r_1+1)\Gamma(q_2+r_1+1),$$

$$t_s = q_1+q_2+\cdots+q_s, \quad \text{and} \quad m_s = r_1+r_2+\cdots+r_s.$$

We then have the following results:

$$I_{2\lambda+1} = \frac{p^{t_{2\lambda+1}}}{2\Gamma(q_1+1)} \prod_{\eta=r_1}^{r_\lambda} \sum_{\eta=0}^{\infty} \frac{(-)^{m_\lambda} p^{2m_\lambda} Q(q_{2\lambda+1}, q_{2\lambda}, 2r_\lambda, 1)}{R(r_\lambda)Q(q_{2\lambda+1}, q_{2\lambda}, r_\lambda, 1)Q(q_{2\lambda+1}, r_\lambda, 1;\, q_{2\lambda}, r_\lambda, 1)}$$

$$\times \frac{\Gamma(\frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+m_\lambda+\frac{1}{4})\Gamma(\frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+m_\lambda+\frac{3}{4})}{\Gamma(\frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+3/2)\Gamma(\frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2})}$$

$$\times {}_2F_3\left[\begin{matrix} \frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+m_\lambda+\frac{1}{4}, & \frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+m_\lambda+\frac{3}{4}; & -p^2 \\ q_1+1, & \frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+\frac{3}{2}, & \frac{1}{2}t_{2\lambda+1}+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2} \end{matrix}\right]. \tag{2.1}$$

In order to establish (2.1), we first consider the integral

$$I_3 \equiv \int_0^1 P_{n+\frac{1}{2}}(x)J_{q_3}(2px)J_{q_2}(2px)J_{q_1}(2px)x^{\nu-\frac{1}{2}}\,dx.$$

We know (Watson) that

$$J_\lambda(z)J_\mu(z) = (z/2)^{\lambda+\mu} \sum_{r=0}^{\infty} \frac{\Gamma(\lambda+\mu+2r+1)(-z^2/4)^r}{\Gamma(r+1)\Gamma(\lambda+\mu+r+1)\Gamma(\lambda+r+1)\Gamma(\mu+r+1)}. \tag{A}$$

Using (A), we have

$$I_3 = \int_0^1 P_{n+\frac{1}{2}}(x) J_{q_1}(2px) \sum_{r_1=0}^{\infty} \frac{(-)^{r_1}\Gamma(q_3+q_2+2r_1+1)(px)^{2r_1+q_3+q_2}}{\Gamma(r_1+1)\Gamma(q_3+q_2+r_1+1)\Gamma(q_3+r_1+1)\Gamma(q_2+r_1+1)} x^{\nu-\frac{1}{2}}\,dx.$$

On changing the order of integration and summation, which is obviously permissible, we get

$$I_3 = p^{q_3+q_2} \sum_{r_1=0}^{\infty} \frac{(-)^{r_1}\Gamma(q_3+q_2+2r_1+1)p^{2r_1}}{\Gamma(r_1+1)\Gamma(q_3+q_2+r_1+1)\Gamma(q_3+r_1+1)\Gamma(q_2+r_1+1)} \times \int_0^1 P_{n+\frac{1}{2}}(x) J_{q_1}(2px) x^{2r_1+q_3+q_2+\nu-\frac{1}{2}}\,dx.$$

If now we use the known integral (Mital, 1941)

$$\int_0^1 P_{n+\frac{1}{2}}(x) J_m(2px) x^{\nu-\frac{1}{2}}\,dx = \frac{p^m\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2})\Gamma(\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4}, & \frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4}; & & -p^2\\ m+1, & \frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2}, & \frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2} & \end{matrix}\right],$$

we get that

$$I_3 = \frac{p^{t_3}}{2\Gamma(q_1+1)} \sum_{r_1=0}^{\infty} \frac{(-)^{r_1}p^{2r_1}\Gamma(q_3+q_2+2r_1+1)}{\Gamma(r_1+1)\Gamma(q_3+q_2+r_1+1)\Gamma(q_3+r_1+1)\Gamma(q_2+r_1+1)}$$
$$\times \frac{\Gamma(\frac{1}{2}t_3+\frac{1}{2}\nu+r_1+\frac{1}{4})\Gamma(\frac{1}{2}t_3+\frac{1}{2}\nu+r_1+\frac{3}{4})}{\Gamma(\frac{1}{2}t_3+\frac{1}{2}\nu+\frac{1}{2}n+r_1+3/2)\Gamma(\frac{1}{2}t_3+\frac{1}{2}\nu-\frac{1}{2}n+r_1+\frac{1}{2})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}t_3+\frac{1}{2}\nu+r_1+\frac{1}{4}, & \frac{1}{2}t_3+\frac{1}{2}\nu+r_1+\frac{3}{4}; & & -p^2\\ q_1+1, & \frac{1}{2}t_3+\frac{1}{2}\nu+\frac{1}{2}n+r_1+\frac{3}{2}, & \frac{1}{2}t_3+\frac{1}{2}\nu-\frac{1}{2}n+r_1+\frac{1}{2} & \end{matrix}\right]. \tag{2.2}$$

By the help of (2.2) and (A), we similarly get that

$$I_5 \equiv \int_0^1 P_{n+\frac{1}{2}}(x) \prod_{\xi=q_1}^{q_5} J_\xi(2px) x^{\nu-\frac{1}{2}}\,dx$$
$$= \frac{p^{t_5}}{2\Gamma(q_1+1)} \sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty} \frac{(-)^{m_2}p^{2m_2}\Gamma(q_5+q_4+2r_2+1)\Gamma(q_3+q_2+2r_1+1)}{\Gamma(r_2+1)\Gamma(r_1+1)\Gamma(q_5+q_4+r_2+1)\Gamma(q_3+q_2+r_1+1)\Gamma(q_5+r_2+1)\Gamma(q_4+r_2+1)}$$
$$\times \frac{\Gamma(\frac{1}{2}t_5+\frac{1}{2}\nu+m_2+\frac{1}{4})\Gamma(\frac{1}{2}t_5+\frac{1}{2}\nu+m_2+\frac{3}{4})}{\Gamma(q_3+r_1+1)\Gamma(q_2+r_1+1)\Gamma(\frac{1}{2}t_5+\frac{1}{2}\nu+\frac{1}{2}n+m_2+\frac{3}{2})\Gamma(\frac{1}{2}t_5+\frac{1}{2}\nu-\frac{1}{2}n+m_2+\frac{1}{2})}$$
$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}t_5+\frac{1}{2}\nu+m_2+\frac{1}{4}, & \frac{1}{2}t_5+\frac{1}{2}\nu+m_2+\frac{3}{4}; & & -p^2\\ q_1+1, & \frac{1}{2}t_5+\frac{1}{2}\nu+\frac{1}{2}n+m_2+\frac{3}{2}, & \frac{1}{2}t_5+\frac{1}{2}\nu-\frac{1}{2}n+m_2+\frac{1}{2} & \end{matrix}\right]. \tag{2.3}$$

Repeating the above process, we arrive at (2.1).

3. Now, we consider m to be even and equal to $2\lambda+2$, say, then

$$I_{2\lambda+2} = \frac{p^{t_{2\lambda+2}}}{2\Gamma(q_2+1)\Gamma(q_1+1)} \prod_{\eta=r_1}^{r_\lambda} \sum_{\eta=0}^{\infty} \frac{(-)^{m_\lambda}p^{2m_\lambda}}{R(r_\lambda)Q(q_{2\lambda+2}, q_{2\lambda+1}, r_\lambda, 1)}$$

$$\times \frac{Q(q_{2\lambda+2}, q_{2\lambda+1}, 2r_\lambda, 1)\Gamma(\frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+m_\lambda+\frac{1}{4})\Gamma(\frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+m_\lambda+\frac{3}{4})}{Q(q_{2\lambda+2}, r_\lambda, 1;\ q_{2\lambda+1}, r_\lambda, 1)\Gamma(\frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+\frac{3}{2})\Gamma(\frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2})}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}q_2+\frac{1}{2}q_1+\frac{1}{2}, \frac{1}{2}q_2+\frac{1}{2}q_1+1, \frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+m_\lambda+\frac{1}{4}, \frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+m_\lambda+\frac{3}{4}; & -4p^2\\ q_1+1,\quad q_2+1,\quad q_2+q_1+1,\quad \frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+\frac{3}{2}, \frac{1}{2}t_{2\lambda+2}+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2} & \end{matrix}\right] \tag{3.1}$$

In order to establish (3.1), we first consider

$$I_4 = \int_0^1 P_{n+\frac{1}{2}}(x)\prod_{\xi=q_1}^{q_4} J_\xi(2px)x^{\nu-\frac{1}{2}}\,dx. \tag{3.2}$$

Proceeding exactly as in I_3 and making use of the integral (Mitai, 1941)

$$\int_0^1 P_{n+\frac{1}{2}}(x)J_l(2px)J_m(2px)x^{\nu-\frac{1}{2}}\,dx$$

$$= \frac{p^{l+m}\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4})}{2\Gamma(l+1)\Gamma(m+1)\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2})\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2})}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}, \frac{1}{2}l+\frac{1}{2}m+1, \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{4}, & \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{3}{4}; & -4p^2\\ l+1,\quad m+1,\quad l+m+1. & \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu+\frac{1}{2}n+\frac{3}{2}, \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2} & \end{matrix}\right]$$

we get

$$I_4 = \frac{p^{t_4}}{2\Gamma(q_1+1)\Gamma(q_2+1)}\sum_{r_1=0}^{\infty}\frac{(-)^{r_1}p^{2r_1}\Gamma(q_4+q_3+2r_1+1)}{\Gamma(r_1+1)\Gamma(q_4+q_3+r_1+1)\Gamma(q_4+r_1+1)\Gamma(q_3+r_1+1)}$$

$$\times\frac{\Gamma(\frac{1}{2}t_4+\frac{1}{2}\nu+r_1+\frac{1}{4})\Gamma(\frac{1}{2}t_4+\frac{1}{2}\nu+r_1+\frac{3}{4})}{\Gamma(\frac{1}{2}t_4+\frac{1}{2}\nu+\frac{1}{2}n+r_1+\frac{3}{2})\Gamma(\frac{1}{2}t_4+\frac{1}{2}\nu-\frac{1}{2}n+r_1+\frac{1}{2})}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{1}{2}q_2+\frac{1}{2}q_1+\frac{1}{2}, \frac{1}{2}q_2+\frac{1}{2}q_1+1, \frac{1}{2}t_4+\frac{1}{2}\nu+r_1+\frac{1}{4}, & \frac{1}{2}t_4+\frac{1}{2}\nu+r_1+\frac{3}{4}; & -4p^2\\ q_1+1,\quad q_2+1,\quad q_2+q_1+1, & \frac{1}{2}t_4+\frac{1}{2}\nu+\frac{1}{2}n+r_1+\frac{3}{2}, \frac{1}{2}t_4+\frac{1}{2}\nu-\frac{1}{2}n+r_1+\frac{1}{2} & \end{matrix}\right]. \tag{3.3}$$

Proceeding further as in §2, we arrive at (3.1).

4. To evaluate

$$I_8' \equiv \int_0^1 P_n(1-2y^2)J_{p_3}(yz)J_{p_2}(yz)J_{p_1}(yz)y^{\nu-1}\,dy$$

we make use of (A) and the integral (Bose, 1946)

$$\int_0^1 P_n(1-2y^2)J_{p_1}(yz)y^{\nu-1}\,dy = \frac{(-)^n z^{p_1}\{\Gamma(\frac{1}{2}p_1+\frac{1}{2}\nu)\}^2}{2^{p_1+1}\Gamma(p_1+1)\Gamma(\frac{1}{2}p_1+\frac{1}{2}\nu-n)\Gamma(\frac{1}{2}p_1+\frac{1}{2}\nu+n+1)}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}p_1+\frac{1}{2}\nu,\quad \frac{1}{2}p_1+\frac{1}{2}\nu; & -z^2/4\\ p_1+1,\quad \frac{1}{2}p_1+\frac{1}{2}\nu-n,\ \frac{1}{2}p_1+\frac{1}{2}\nu+n+1 & \end{matrix}\right],$$

$p_1+\nu>0$, which gives

$$I_8' = \frac{(z/2)^{p_1+p_2+p_3}}{2\Gamma(p_1+1)}\sum_{r_1=0}^{\infty}\frac{(-)^{n+r_1}(\frac{1}{2}z)^{2r_1}\Gamma(p_3+p_2+2r_1+1)}{\Gamma(r_1+1)\Gamma(p_3+p_2+r_1+1)\Gamma(p_3+r_1+1)\Gamma(p_2+r_1+1)}$$

$$\times\frac{\{\Gamma(\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+r_1)\}^2}{\Gamma(\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu-n+r_1)\Gamma(\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+n+r_1+1)}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+r_1,\ \frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+r_1; & -z^2/4\\ p_1+1,\quad \frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu-n+r_1; \frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+n+r_1+1 & \end{matrix}\right] \tag{4.1}$$

We can then evaluate I_4', by expanding $J_{p_4}(yz)$ and after changing the order of integration, and using (4.1), we get that

$$I_4' = \frac{(z/2)^{p_1+p_2+p_3+p_4}}{2\Gamma(p_1+1)} \sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty} \frac{(-)^{n+m_2}\Gamma(p_3+p_2+2r_1+1)(\frac{1}{2}z)^{2m_2}}{\Gamma(r_2+1)\Gamma(r_1+1)\Gamma(p_4+r_2+1)\Gamma(p_3+p_2+r_1+1)}$$

$$\times \frac{\{\Gamma(\frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+m_2)\}^2}{\Gamma(p_3+r_1+1)\Gamma(p_2+r_1+1)\Gamma(\frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu-n+m_2)}$$

$$\times \frac{1}{\{\Gamma(\frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+n+m_2+1)\}^2}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+m_2,\ \frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+m_2\,; \\ p_1+1,\ \frac{1}{2}p_1+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu-n+m_2,\ \frac{1}{2}p_4+\frac{1}{2}p_3+\frac{1}{2}p_2+\frac{1}{2}p_1+\frac{1}{2}\nu+n+m_2+1\end{matrix}\ -z^2/4\right] \quad (4.2)$$

Further we can write (4.2) as

$$I_4' = (\tfrac{1}{2}z)^{p_4} \sum_{r_2=0}^{\infty} \frac{(-)^{r_2}(\frac{1}{2}z)^{2r_2}}{\Gamma(r_2+1)\Gamma(r_2+p_4+1)} I_3'$$

when for ν in I_3', we write $\nu+2r_2+p_4$. On repeating this process we arrive at (1.1).

5. *Particular cases*: (*i*) From (2.1) and (3.1), it is possible to write the values of the following integrals:

$$(a)\quad \int_0^1 P_{n+\frac{1}{2}}(x)\sin^m(2px)x^{\nu-\frac{1}{2}}dx;$$

$$(b)\quad \int_0^1 P_{n+\frac{1}{2}}(x)\cos^m(2px)x^{\nu-\frac{1}{2}}dx.$$

(*ii*) If $q_1 = q_2 = \cdots = q_{2\lambda+1} = q$, say, then (2.1) reduces to

$$I_{2\lambda+1} \equiv \int_0^1 P_{n+\frac{1}{2}}(x)\{J_q(2px)\}^{2\lambda+1}x^{\nu-\frac{1}{2}}dx = \frac{2^{2\lambda q-1}p^{(2\lambda+1)q}}{\pi^{\lambda/2}\Gamma(q+1)} \prod_{\eta=r_1}^{r_\lambda}\sum_{\eta=0}^{\infty} \frac{(-)^{m_\lambda}p^{2m_\lambda}Q(q,r_\lambda,\frac{1}{2})}{R(r_\lambda)Q(2q,r_\lambda,1)}$$

$$\times \frac{2^{2m_\lambda}\Gamma\{\frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+m_\lambda+\frac{1}{4}\}\Gamma\{\frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+m_\lambda+\frac{3}{4}\}}{Q(q,r_\lambda,1)\Gamma\{\frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+\frac{3}{2}\}\Gamma\{\frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2}\}}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+m_\lambda+\frac{1}{4}, & \frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+m_\lambda+\frac{3}{4}\,; & -p^2 \\ q+1, & \frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu+\frac{1}{2}n+m_\lambda+\frac{3}{2}, & \frac{1}{2}(2\lambda+1)q+\frac{1}{2}\nu-\frac{1}{2}n+m_\lambda+\frac{1}{2}\end{matrix}\right].$$

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THE MOMENTS AND SEMINVARIANTS OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE

By

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The object of this note is to obtain by making use of the characteristic function, elegant expressions for the general moments and seminvariants of the distribution law of the statistic

$$\delta^2 = \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2/(n-1)$$

where $x_1, x_2, \ldots, x_n$ is a random sample of size n from a normal universe whose mean is zero and standard deviation is σ. Jointly Von Neumann, Kent, Bellinson and Hart (1941) first obtained the moments by using a different method.

1. Let us start by proving the following lemma which will be found useful in what follows. I have followed Williams (1941, p. 240) and Von Neumann (1941, p. 371) in establishing this.

Lemma. Let $n > 2$. The value of the n-rowed determinant

$$K_n(x) = \begin{vmatrix} 1-x & x & 0 & 0 & \ldots & 0 & 0 & 0 \\ x & 1-2x & x & 0 & \ldots & 0 & 0 & 0 \\ 0 & x & 1-2x & x & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & x & 1-2x & x \\ 0 & 0 & 0 & 0 & \ldots & 0 & x & 1-x \end{vmatrix} \tag{1}$$

is equal to

$$\sum_{j=0}^{n-1} (-1)^j \binom{2n-j-1}{j} x^j \tag{2}$$

and the zeros of (2) are the numbers

$$1/4 \sin^2 (\pi\mu/2n) \tag{3}$$

where $\mu = 1, 2, 3, \ldots, n-1$.

Proof: Writing μ_n for the determinant obtained from (1) by changing only the first element from $1-x$ to $1-2x$, we can easily deduce $\mu_n = (1-2x)\mu_{n-1} - x^2\mu_{n-2}$ and $K_n = \mu_n + x\mu_{n-1}$. It is easy to prove by induction that

$$\mu_n = \sum_{r=0}^{n} (-1)^r \binom{2n-r}{r} x^r.$$

and hence on account of the relationship between μ_n and K_n, we get easily

$$K_n(x) = \sum_{r=0}^{n-1} (-1)^r \binom{2n-1-r}{r} x^r.$$

Any value of x for which the system of linear equations

$$x_{r-1}+x_{r+1} = (2-1/x)x_r, \quad r = 1, 2, \ldots, n,$$

where $x_0 = x_1$ and $x_n = x_{n+1}$, will have a set of solutions not all zero, will make the value of the determinant (1) zero. It will be observed that for

$$x = 1/4 \sin^2 (\mu\pi/2n) \qquad (\mu = 1, 2, 3, \ldots, n-1),$$
$$x_r = 2 \cos \{(r-\tfrac{1}{2})\mu\pi/n\} \quad (r = 1, 2, \ldots, n)$$

will satisfy the above equations and of these at least

$$x_1 = 2 \cos (\mu\pi/2n) \neq 0 \text{ for } \mu = 1, 2, \ldots, n-1.$$

Hence, any one of the $(n-1)$ distinct positive numbers $1/4 \sin^2(\mu\pi/2n)$ $(\mu = 1, 2, \ldots, n-1)$ will make (1) zero and since (1) is a polynomial of the $(n-1)$-th degree, it can not have any other zero.

2. *Characteristic function of the distribution law of* δ^2. Here

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^n} \exp \{-\sum_{i=1}^{n} x_i^2/2\sigma^2\}.$$

And the characteristic function of the distribution law of δ^2

$$\varphi(t) = \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp(it\delta^2).\exp\left\{-\sum_{i=1}^{n} x_i^2/2\sigma^2\right\} dx_1 dx_2 \ldots dx_n$$

$$= \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^n} \cdot \frac{(\sqrt{\pi})^n}{\begin{vmatrix} \frac{1}{2\sigma^2}-\frac{it}{n-1}, & \frac{it}{n-1}, & 0, & 0, & \ldots & 0 \\ \frac{it}{n-1}, & \frac{1}{2\sigma^2}-\frac{2it}{n-1}, & \frac{it}{n-1}, & 0, & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}^{\frac{1}{2}}}$$

$$= 1\Big/\left\{\sum_{j=0}^{n-1} (-1)^j \binom{2n-j-1}{j}\left(\frac{2\sigma^2 it}{n-1}\right)^j\right\}^{\frac{1}{2}} \tag{4}$$

by (2). We have also

$$\varphi(t) = 1\Big/\left\{\prod_{\mu=1}^{n-1}\left[1-\frac{it\sigma^2}{n-1}\cdot 8 \sin^2 \frac{\pi\mu}{2n}\right]\right\}^{\frac{1}{2}} \tag{5}$$

by (3).

3. *Seminvariants of the distribution law of* δ^2. We have from (5)

$$\log \varphi(t) = \frac{1}{2}\sum_{\mu=1}^{n-1}\sum_{k=1}^{\infty} \frac{\sigma^{2k}8^k \sin^{2k}(\pi\mu/2n)}{(n-1)^k}\frac{(it)^k}{k} = \sum_{k=1}^{\infty} \lambda_k \frac{(it)^k}{k!}$$

where λ_k denotes the k-th seminvariant. Hence

$$\lambda_k = \frac{2^{k-1}\sigma^{2k}(k-1)!}{(n-1)^k}\sum_{\mu=1}^{n-1}\left[4\sin^2\frac{\pi\mu}{2n}\right]^k = \frac{2^{k-1}\sigma^{2k}(k-1)!}{(n-1)^k}\,s_k \tag{6}$$

where s_k denotes the sum of the kth powers of the roots of the equation

$$\sum_{j=0}^{n-1}(-1)^j\binom{2n-1-j}{j}x^{n-1-j}=0. \tag{7}$$

Hence λ_k can be obtained in the form of a determinant whose elements are obtained from the coefficients of the polynomial on the left hand side of (7). Thus

$$\lambda_1 = 2\sigma^2,$$

$$\lambda_2 = \frac{2\sigma^4}{(n-1)^2}\begin{vmatrix} -(2n-2) & 1 \\ 2\binom{2n-3}{2} & -(2n-2)\end{vmatrix} = \frac{4\sigma^4}{(n-1)^2}(3n-4),$$

$$\lambda_3 = -\frac{4.2!.\sigma^6}{(n-1)^3}\begin{vmatrix} -(2n-2) & 1 & 0 \\ 2\binom{2n-3}{2} & -(2n-2) & 1 \\ -3\binom{2n-4}{3} & \binom{2n-3}{2} & -(2n-2)\end{vmatrix} = \frac{32\sigma^6}{(n-1)^3}(5n-8),$$

$$\lambda_4 = \frac{8.3!.\sigma^8}{(n-1)^4}\begin{vmatrix} -(2n-2) & 1 & 0 & 0 \\ 2\binom{2n-3}{2} & -(2n-2) & 1 & 0 \\ -3\binom{2n-4}{3} & \binom{2n-3}{2} & -(2n-2) & 1 \\ 4\binom{2n-5}{4} & -\binom{2n-4}{3} & \binom{2n-3}{2} & -(2n-2)\end{vmatrix} = \frac{96\sigma^8}{(n-1)^4}(35n-64),$$

and so on. Alternatively we can also write down the kth seminvariant in terms of the seminvariants of lower order from the following formula, which can be easily deduced from known relations connecting sums of powers of zeros of a polynomial with the coefficients of the polynomial:

$$\lambda_k = \begin{cases} -\displaystyle\sum_{r=1}^{k-1}(2n-2r)_r\binom{k-1}{r}\left(\frac{-2\sigma^2}{n-1}\right)^r\lambda_{k-r} - \frac{1}{2}\left(\frac{-2\sigma^2}{n-1}\right)^k(2n-2k)_k, & \text{if } k\le n-1, \\[2ex] -\displaystyle\sum_{r=1}^{n-1}(2n-2r)_r\binom{k-1}{r}\left(-\frac{2\sigma^2}{n-1}\right)^r\lambda_{k-r}, & \text{if } k> n-1. \end{cases}$$

In the above formula, the symbol $(a)_k$ stands for $a(a+1)\ldots(a+k-1)$.

4. *The moments of the distribution law of* δ^2. From (4), the moment generating function

$$\psi(t) = \sum_{k=0}^{\infty} \mu'_k \frac{t^k}{k!} = \frac{1}{[p(t)]^{\frac{1}{2}}},$$

where

$$p(t) = \sum_{j=0}^{n-1} (-1)^j \binom{2n-j-1}{j} \left(\frac{2\sigma^2 t}{n-1}\right)^j.$$

If

$$\psi^{(s)}(t) = \frac{\mu_s'(t)}{[p(t)]^{(2s+1)/2}}, \quad \psi^{(s+1)}(t) = \frac{\dfrac{d\mu_s'(t)}{dt}\, p(t) - \dfrac{2s+1}{2}\, \mu_s'(t) p'(t)}{[p(t)]^{(2s+3)/2}}$$

Therefore writing $\mu_1'(t) = -\frac{1}{2} p'(t)$ and in general

$$\mu'_{s+1}(t) = \frac{d\mu_s'(t)}{dt}\, p(t) - \frac{2s+1}{2}\, \mu_s'(t) p'(t)$$

$s = 1, 2, 3, \ldots,$ we get easily the recursion formula

$$\mu'_{s+1} = \mu'_{s+1}(0) = \left[\frac{d\mu_s'(t)}{dt}\, p(t) - \frac{2s+1}{2}\, \mu_s'(t) p'(t)\right]_{t=0} = 2(2s+1)\mu_s' + \left[\frac{d\mu_s'(t)}{dt}\right]_{t=0}.$$

where μ_s' denotes the s-th moment of δ^2 about the origin. Thus

$$\mu_1'(t) = -\tfrac{1}{2} p'(t)$$

$$\mu_2'(t) = -\tfrac{1}{2}[p''(t)p(t) - \tfrac{3}{2}\{p'(t)\}^2]$$

$$\mu_3'(t) = -\tfrac{1}{2}[p'''(t)\{p(t)\}^2 - \tfrac{9}{2} p''(t)p'(t)p(t) + \tfrac{15}{4}\{p'(t)\}^3]$$

$$\mu_4'(t) = -\tfrac{1}{2}[p^{iv}(t)\{p(t)\}^3 - 6p'''(t)p'(t)\{p(t)\}^2 - \tfrac{9}{2}\{p''(t)\}^2\{p(t)\}^2 + \tfrac{45}{2} p''(t)\{p'(t)\}^2 p(t) - \tfrac{105}{8}\{p'(t)\}^4]$$

and so on. Remembering that

$$p^k(0) = \begin{cases} (-1)^k \dbinom{2n-k-1}{k} \left(\dfrac{2\sigma^2}{n-1}\right)^k . k!, & \text{if } 1 \leq k \leq n-1, \\ 0 & \text{if } \quad k > n-1, \end{cases}$$

we get easily for $n \geq 5$

$$\mu_1' = 2\sigma^2$$

$$\mu_2' = -\frac{1}{2}\left[\frac{(2n-3)(2n-4)}{(n-1)^2} \cdot 4\sigma^4 - \tfrac{3}{2}.16.\sigma^4\right] = \frac{4\sigma^4}{(n-1)^2}(n^2+n-3)$$

$$\mu_3' = -\frac{1}{2}\left[-\frac{(2n-4)(2n-5)(2n-6)}{(n-1)^3} \cdot 8\sigma^6 + \frac{9}{2}\, \frac{(2n-3)(2n-4)}{(n-1)^2} \cdot 16\sigma^6 - \frac{15}{4} \cdot 4^3\sigma^6\right]$$

$$= \frac{8\sigma^6}{(n-1)^3}(n^3 + 6n^2 + 2n - 21)$$

$$\mu_4' = -\frac{1}{2}\left[\frac{(2n-5)(2n-6)(2n-7)(2n-8)}{(n-1)^4}\,16\sigma^8 - 6.4\sigma^2\,\frac{(2n-4)(2n-5)(2n-6)}{(n-1)^3}\,8\sigma^6\right.$$

$$\left.-\frac{9}{2}\,\frac{(2n-3)^2(2n-4)^2}{(n-1)^4}\,16\sigma^3 + \frac{45}{2}\cdot 16\sigma^4\,\frac{(2n-3)(2n-4)}{(n-1)^2}\cdot 4\sigma^4 - \frac{105}{8}\cdot 256\sigma^8\right]$$

$$= \frac{16\sigma^8}{(n-1)^4}\,(n^4 + 14n^3 + 58n^2 - 8n - 231).$$

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NOTE ON SYMMETRICAL INCOMPLETE BLOCK DESIGNS: $\lambda = 2$, $k = 6$ or 7

By

S. M. Kerawala

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The purpose of the present paper is to establish the results of Husain's (1945, 1946) two papers in a more rapid and elegant manner.

The design $v = b = 16$, $r = k = 6$, $\lambda = 2$

The initial block contains varieties 1 to 6, and the other 15 blocks are denoted by the 15 pairs of these initial varieties. The blocks in which the varieties 7 to 16 are placed have been represented by Husain by two types of chaining, *viz.*, $A = (123456)$ and $B = (123)\ (456)$.

We assume that at least one chain of type A occurs in the set of ten mutually consistent chains to be found. This may be taken to be (123456) and will be denoted by [0, 0]. The totality of chains consistent with this standard chain may be represented as below:

COMMON BLOCKS →	(12), (34)	(23), (45)	(34), (56)	(45), (61)	(56), (12)	(61), (23)	(12), (45)	(23), (56)	(34), (61)	← COMMON BLOCKS
COLUMNS →	1	2	3	4	5	6	7	8	9	↓ ROWS
1	(126435)	(132645)	(134265)	(135426)	(124653)	(153246)	(126453)	(132465)	(153426)	1
2	(125) (346)	(145) (236)	(134) (256)	(136) (245)	(124) (356)	(146) (235)	(126354)	(132564)	(152436)	2
							(125163)	(142365)	(143526)	3

The chains have been tabulated according to the pairs of blocks which they have in common with [0, 0], and the manner of their construction becomes obvious from the fact that columns 2 to 6 are derived from column 1, and the columns 8 and 9 from column 7 by transformations which leave [0, 0] unchanged. $[r, s]$ will denote the cycle in the r-th row and the s-th column. To solve the problem, we must select one cycle from each column in such a manner that the nine cycles selected are mutually consistent. Starting off with column 1 and pursuing the chains to successive columns, watching only for mutual consistence, we readily arrive at the following sets:

[0, 0], [1, 1], [2, 2], [2, 3], [1, 4], [2, 5], [2, 6], [3, 7], [1, 8] and [2, 9];
[0, 0], [2, 1], [1, 2], [2, 3], [2, 4], [1, 5], [2, 6], [2, 7], [3, 8] and [1, 9];
[0, 0], [2, 1], [2, 2], [1, 3], [2, 4], [2, 5], [1, 6], [1, 7], [2, 8] and [3, 9];
[0, 0], [2, 1], [2, 2], [2, 3], [2, 4], [2, 5], [2, 6], [1, 7], [1, 8] and [1, 9].

The first three solutions consist of four chains of type B and six of type A; they are isomorphic, as any two become identical on applying a suitable transformation which transforms [0, 0] into itself. The fourth solution consists of four chains of type A and six of type B, and is self-conjugate. Hence the four solutions reduce to two independent ones.

It is, however possible that no chain of type A may occur in a solution. In that case, all the ten chains would be of type B. Taking (123)(456) as the incipient chain, there exist only nine chains of type B consistent with it. These are (124)(356), (125)(346), (126)(345), (134)(256), (135)(246), (136)(245), (145)(236), (146)(235), (156)(234). On examination, these are found to be mutually consistent and hence provide a solution, which proves to be a self-conjugate one.

Hence there exist only three independent solutions altogether, of which two are self-conjugate.

THE DESIGN $v = b = 22,\ r = k = 7,\ \lambda = 2$

Again the varieties in the initial block are 1 to 7 and the other 21 blocks are denoted by the 21 pairs of these initial numbers. Only two types of chaining are possible, *viz.*, A = (1234567) and B = (123)(4567) or (1234)(567). If we assume that a solution exists in which at least one chain of type B occurs, we can make (123)(4567) the incipient chain. On looking for chains consistent with this and having common with it the blocks (45), (67), we fail to find any. Hence no chain of type B can enter into a solution, which will thus contain only chains of type A. Let the incipient chain [0, 0] now be (1234567). Chains of type A consistent with [0, 0] may be tabulated as below:

COMMON BLOCKS→	(12), (34)	(23), (45)	(34), (56)	(45), (67)		←C. B.
COLUMNS→	1	2	3	4		↓ROWS
1	(1275346)	(1327546)	(1342756)	(1354276)		1
2	(1257436)	(1547236)	(1347265)	(1376245)		2
3	(1257346)	(1457236)	(1347256)	(1367245)		3
4	(1264375)	(1457326)	(1437256)	(1367254)		4
5	(1263475)	(1547326)	(1437265)	(1376254)		5

Again the chains of any row in columns 2 to 4 have been derived from the chain of the same row in column 1 by a transformation which transforms [0, 0] into itself. To obtain a solution 14 chains, one from each column and mutually consistent, are to be selected. We begin separately with the different chains in column 1, and pursue them—keeping an eye on consistence—to cycles in other columns.

[1, 1]' is inconsistent with any cycle in column 4. Hence no chain in row 1 can enter into a solution. The table may, therefore, be modified by dropping row 1. Again, [2, 1] is inconsistent with any chain in column 3. Hence the second row

may be scratched out. Now [3, 1] is inconsistent with the remaining cycles in column 3, and so the third row becomes superfluous. Further, [4, 1] is inconsistent with either [4, 3] or [5, 3], and so the fourth row drops out. Finally, [5, 1] is inconsistent with [5, 2], and hence there is no solution.

The impossibility of a solution for $k = 7$ is thus established.

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ON RAMANUJAN'S FUNCTION $\tau(n)$ AND THE DIVISOR FUNCTION $\sigma_k(n)$—I

By

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1. In this paper we shall develop a systematic method of studying certain congruence properties of the divisor function, $\sigma_k(n)$, the sum of the k-th powers of the divisors of n. The type of relation considered is exemplified by the following :-

$$\sigma_{13}(n) \equiv 11\sigma_9(n)+22\sigma_7(n)-32\sigma_3(n) \qquad (\text{mod } 2^7.3^3.5.7.11),$$

$$\sigma_{13}(n) \equiv 380\sigma_7(n)-21(40n-21)\sigma_5(n)+20\sigma(n) \quad (\text{mod } 2^6.3^3.5.7^2).$$

A result which has been used in this paper in an auxiliary capacity, although it has its own interest, is Theorem A. This may be stated in the following alternative form.

Theorem A[1].

$$n^u\sigma_{v-u}(n)+n^r\sigma_{s-r}(n) \equiv n^u\sigma_{s-u}(n)+n^r\sigma_{v-r}(n) \quad (\text{mod } g)$$

where $g\,|\,(n^u-n^r)(n^v-n^s)$ *for all positive integral values of* n*;* u, v, r, s *being given positive integers.*

It is also worth noting its

Corollary :

$$n^k(1+n^b)\sigma_a(n) \equiv n^k(1+n^a)\sigma_b(n) \quad (\text{mod } g)$$

where $g\,|\,(n^{a+k}-n^{b+k})(n^{a+b+k}-n^k)$ for all positive integral values of n; a, b, k being given positive integers.

This is useful in studying the roots of the congruence

$$\sigma_a(n) \equiv 0 \quad (\text{mod } g).$$

In particular we get without much trouble Ramanathan's (1945*a*) result:

If n *is a non-residue of the prime* p *then*

$$\sigma_{(p-1)/2}(n) \equiv 0 \quad (\text{mod } p).$$

We shall also show in subsequent parts of this paper how our treatment can be extended, and it is indeed here that the main interest of the method lies, to the derivation of the congruence properties of Ramanujan's Function $\tau(n)$, which as is well-known, is defined by

$$x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24} = \sum_1^\infty \tau(n)x^n, \quad |x|<1.$$

As illustrations of the type of results we get the following may be stated

$$\tau(n) \equiv n^2\sigma_7(n) - 9n^2\sigma_5(n) + 9n^3\sigma_3(n) \pmod{2^4.3^4.5},$$

$$199584\tau(n) \equiv 286650\sigma_{11}(n) - 691[231(6n-5)\sigma_9(n) + 1100\sigma_7(n)$$
$$-1155(2n-1)\sigma_5(n) - 50\sigma(n)] \pmod{2^7.3^3.5^2.7.11.691}.$$

These results, 171 in number, have been set down in Table C(2) of Part II of this paper, and the corresponding results involving $\sigma_k(n)$'s only are given in Table C(1). The method can be further extended to obtain a larger number of congruence relations involving $\tau(n)$. These results will be published in Part III.

Results of this type due to Ramanujan are

$$\tau(n) \equiv n\sigma(n) \pmod{5},$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Gupta (1945a) has shown that

$$\tau(n) \equiv \sigma(n) \pmod{j},$$

if $(n, j) = 1$ and $j|24$. The author (Lahiri, 1946a) has obtained substantially the same result. He has determined the residues of $\tau(n)$ to moduli $j|24$. Residues to moduli 3 and 4 have also been obtained by a different process by Ramanathan (1945a). He also obtains Wilton's results (1930) for modulus 7 and proves incidentally that

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}.$$

It may be pointed out here that since $\sigma_k(n)$ is a comparatively simple function to deal with it is fairly simple to determine the residues of $\tau(n)$ to modulus j if a congruence relation, with respect to the same modulus, connecting $\tau(n)$ with $\sigma_k(n)$'s is known. I understand Chowla and Bambah have obtained some new congruence relations connecting $\tau(n)$ and $\sigma_k(n)$.

We shall prove incidentally a large number of identities of the type

$$12[\sum_1^\infty n\sigma(n)x^n]^2 = \sum_1^\infty [n^2\sigma_3(n) - n^3\sigma(n)]x^n,$$

$$12096[\sum_1^\infty \sigma_3(n)x^n]^2.[\sum_1^\infty \sigma_5(n)x^n] = \sum_1^\infty [\sigma_{13}(n) - 22\sigma_9(n) + 20\sigma_7(n) + 21\sigma_5(n) - 20\sigma_3(n)]x^n,$$

$$840[\sum_1^\infty n^2\sigma(n)x^n]^2 = \sum_1^\infty [15n^4\sigma_3(n) - 14n^5\sigma(n) - \tau(n)]x^n,$$

$$640[\sum_1^\infty n\sigma_3(n)x^n]^2 = \sum_1^\infty [n^2\sigma_7(n) - \tau(n)]x^n,$$

$$40320[\sum_1^\infty n\sigma(n)x^n]^3 = \sum_1^\infty [85n^3\sigma_5(n) - 120n^4\sigma_3(n) + 84n^5\sigma(n) + \tau(n)]x^n.$$

Results equivalent to this type have been set down in Tables B(1) of this part and in B(2) of the second part of this paper. The notation used there will be explained in the next section.

Analogous identities, in a sense simpler although more fundamental in character, have been obtained by Ramanujan (1916) in an equivalent form in his Table IV,

We have also indicated how it is possible to extract from the congruences referred to above congruence relationships of the type

$$A\tau(n) \equiv 0, \qquad A\tau(n) \equiv B\sigma_k(n),$$

as also

$$A\sigma_k(n) \equiv 0, \qquad A\sigma_k(n) \equiv B\sigma_l(n),$$

where A's and B's are polynomials in n, with respect to suitable moduli. To mention just two we have

$$n^5(n-1)^3(n-2)\tau(n) \equiv 0 \pmod{2^5.3^3},$$

$$\tau(n) \equiv -n^3(6n-7)\sigma_3(n) \pmod{2^6.3.7}.$$

These are helpful in connection with the problem of determining linear forms $am+b$ such that

$$\tau(am+b) \equiv 0, \quad \sigma_k(am+b) \equiv 0,$$

for certain moduli. Such forms were given by Ramanujan (1920), *viz.*,

$$\tau(5m) \equiv 0 \pmod 5,$$

$$\tau(7m) \equiv 0 \pmod 7,$$

$$\tau(am+b) \equiv 0 \pmod a, \quad a = 7, 23,$$

b being any quadratic non-residue of a.

Banerji (1942), Gupta (1943) and Ramanathan (1944) have obtained other forms. Such linear forms relating the function $\sigma_k(n)$ have also been studied by the last two authors [Ramanathan (1943, 1945*b*), Gupta (1945*b*)].

It also follows from the congruence relations established in the first two parts of this paper the following extention of a result due to Walfisz:

$\tau(n)$ is divisible by $2^{10}.3^5.5^3.7.691$ for almost all values of n.

Finally it may be pointed out that the treatment throughout this paper is elementary.

2. Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n, and let

$$\sigma_k(0) = \tfrac{1}{2}\zeta(-k),$$

where $\zeta(k)$ is the Riemann Zeta-function. Further let

(1) $$\Sigma_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0).$$

Then Ramanujan (1916) proved that

(2) $$\Sigma_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\cdot\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\,\sigma_{r+s+1}(n) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}.n\sigma_{r+s-1}(n),$$

for the following nine pairs of values of r and s: $r=1, s=1$; $r=1, s=3$; $r=1, s=5$; $r=1, s=7$; $r=1, s=11$; $r=3, s=3$; $r=3, s=5$; $r=3, s=9$; $r=5, s=7$.

We shall now show how the above identity can be expressed in an equivalent but more suggestive form by making use of another function $\Phi_{r,s}(x)$ also used by Ramanujan, and defined by

$$\Phi_{r,s}(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn} = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n. \tag{3}$$

We shall however find it convenient to change Ramanujan's notation and shall write simply (r, s) for $\Phi_{r,s}(x)$.

It can be easily seen that in virtue of relation (2)

$$\sum_{n=0}^{\infty} \Sigma_{r,s}(n)x^n = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\cdot\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\{\tfrac{1}{2}\zeta(-r-s-1)+(0,\, r+s+1)\} + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}.(1,\, r+s), \tag{4}$$

for the nine pairs of values of r and s. It is also not difficult to see that

$$\sum_{n=0}^{\infty} \sigma_r(n)x^n . \sum_{n=0}^{\infty} \sigma_s(n)x^n = \tfrac{1}{4}\zeta(-r)\zeta(-s)+\tfrac{1}{2}\zeta(-s).(0,\, r)+\tfrac{1}{2}\zeta(-r).(0,\, s)+(0,\, r)(0,\, s). \tag{5}$$

But relation (1) shows that

$$\sum_{n=0}^{\infty} \sigma_r(n)x^n . \sum_{n=0}^{\infty} \sigma_s(n)x^n = \sum_{n=0}^{\infty} \Sigma_{r,s}(n)x^n.$$

It follows therefore from (4) and (5) that

$$(0,\, r)(0,\, s) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\cdot\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}.(0,\, r+s+1) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}.(1,\, r+s) - \tfrac{1}{2}\zeta(-s)(0,\, r) - \tfrac{1}{2}\zeta(-r)(0,\, s), \tag{6}$$

for the nine pairs of values of r and s.

Thus Ramanujan's identity (2) when viewed through (6) shows that for certain values of v's the product of two $(0,\, v)$'s can be expressed as a linear expression in $(u,\, v)$'s. Now it is hardly necessary to point out that the following problem is naturally suggested—to determine all products of two or more $(u,\, v)$'s which can be expressed as a linear function of $(u,\, v)$'s. It will be our purpose to determine all such products as given by a method which is explained below.

Let

$$P = 1 - 24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\cdots\right), \tag{7}$$

$$Q = 1 + 240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\cdots\right), \tag{8}$$

$$R = 1 - 504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\cdots\right), \tag{9}$$

then Ramanujan proved by elementary methods that when $r+s$ is odd, (r, s) is expressible as a polynomial in P, Q and R, in the form

$$(r, s) = \sum k_{l,m,n} P^l Q^m R^n, \tag{10}$$

where $l-1 \leq \min(r, s)$ $2l+4m+6n = r+s+1$. Polynomial expressions for some (r, s)'s are given in Tables I, II and III of Ramanujan's paper. An immediate deduction of the above result is that the product of any finite number of (r, s)'s, $r+s$ odd, can be expressed as a polynomial in P, Q and R. If each term $P^l.Q^m.R^n$ entering in the polynomial expression can be expressed as a linear function of (u, v)'s it would follow that the product is also expressible as such.

It appears therefore that our first step should be the determination of all products $P^l.Q^m.R^n$ which are expressible as a linear function of (u, v)'s. A list of such products is given in the following Table. These results are easily derivable from Ramanujan's Tables I, II and III. As an example the formula for P^3 is obtained by a process of elimination from the identities III 1, II 2 and I 3 of Ramanujan, and this is indicated on the extreme right hand side of formula 3 in the Table

TABLE A(1)

1.	$P = 1-2^3.3(0, 1)$,	[I 1
2.	$P^2 = 1-2^6.3^2(1, 2)+2^4.3.5(0, 3)$,	[II 1, I 2
3.	$P^3 = 1-2^6.3^3(2, 3)+2^4.3^3.5(1, 4)-2^3.3^2.7(0, 5)$,	[III 1, II 2, I 3
4.	$P^4 = 1-2^8.3^3(3, 4)+2^7.3^4(2, 5)-2^6.3^2.7(1, 6)$ $+2^5.3.5(0, 7)$,	[III 6, 2, II 3, I 4
5.	$P^5 = 1-2^5.3^4(4, 5)+2^8.3^3.5(3, 6)-2^7.3^3.5(2, 7)$ $+2^4.3^3.5^2(1, 8)-2^3.3\ 11(0, 9)$,	[III 9, 7, 3, II 4, I 5
6.	$Q = 1+2^4.3.5(0, 3)$,	[I 2
7.	$Q^2 = 1+2^5.3.5(0, 7)$,	[I 4
8.	$R = 1-2^3.3^2.7(0, 5)$,	[I 3
9.	$PQ = 1+2^4.3^2.5(1, 4)-2^3.3^2.7(0, 5)$,	[II 2, I 3
10.	$P^2Q = 1+2^6.3^3(2, 5)-2^5.3^2.7(1, 6)+2^5.3.5(0, 7)$,	[III 2, II 3, I 4
11.	$P^3Q = 1+2^7.3^3(3, 6)-2^6.3^4(2, 7)+2^4.3^3.5(1, 8)$ $-2^3.3.11(0, 9)$,	[III 7, 3, II 4, I 5
12.	$PQ^2 = 1+2^4.3^2.5(1, 8)-2^3.3.11(0, 9)$,	[II 4, I 5
13.	$PR = 1-2^4.3^2.7(1, 6)+2^5.3.5(0, 7)$,	[II 3, I 4
14.	$P^2R = 1-2^6.3^3(2, 7)+2^5.3^2.5(1, 8)-2^3.3.11(0, 9)$,	[III 3, II 4, I 5
15.	$QR = 1-2^3.3.11(0, 9)$,	[I 5
16.	$Q^2R = 1-2^3.3(0, 13)$.	[I 7

3. This completes the first step. The next step consists in the enumeration of all products of (u, v)'s whose polynomial expressions involve only terms $P^l.Q^m.R^n$ of the forms given in the above Table. The final step is the substitution of the linear expressions as given in Table A(1) for these terms. We thus get the following Table B(1). It may be pointed out however that in the actual derivation we have adopted certain simplifying artifices. For example, a judicious use of the fact

$$x\frac{d}{dx}(u, v) = (u+1, v+1), \tag{11}$$

sometimes helps to reduce the labour of derivation of the results.

TABLE B(1)

(3.1)	$2^2.3(0, 1)^2$	$= 5(0, 3)-2.3(1, 2)+(0, 1),$
(3.2)	$2^3.3(0, 1)(1, 2)$	$= 5(1, 4)-2.3(2, 3)+(1, 2),$
(3.3)	$2^2.3(1, 2)^2$	$= (2, 5)-(3, 4),$
(3.4)	$2^3.3(0, 1)(2, 3)$	$= 3(2, 5)-2^2(3, 4)+(2, 3),$
(3.5)	$2^3.3(1, 2)(2, 3)$	$= (3, 6)-(4, 5),$
(3.6)	$2^3.3(0, 1)(3, 4)$	$= 2(3, 6)-3(4, 5)+(3, 4);$
(5.1)	$2^4.3.5(0, 1)(0, 3)$	$= 3.7(0, 5)-2.3.5(1, 4)+2.5(0, 3)-(0, 1),$
(5.2)	$2^6.3(0, 1)^3$	$= 7(0, 5)-2.3.5(1, 4)+2.5(0, 3)+2^3.3(2,3)-2^2.3(1, 2)+(0, 1),$
(5.3)	$2^3.3.5(0, 1)(1, 4)$	$= 7(1, 6)-2^2.3(2, 5)+5(1, 4),$
(5.4)	$2^4.3.5(1, 2)(0, 3)$	$= 7(1, 6)-2.3(2, 5)-(1, 2),$
(5.5)	$2^6.3^2(0, 1)^2(1, 2)$	$= 7(1, 6)-2.3.5(2, 5)+2.5(1, 4)+2^3.3(3, 4)-2^2.3(2, 3)+(1, 2),$
(5.6)	$2^2.3.5(1, 2)(1, 4)$	$= (2, 7)-(3, 6),$
(5.7)	$2^3.3(0, 1)(2, 5)$	$= (2, 7)-2(3, 6)+(2, 5),$
(5.8)	$2^4.3.5(2, 3)(0, 3)$	$= 3(2, 7)-2(3, 6)-(2, 3),$
(5.9)	$2^5.3^2(0, 1)(1, 2)^2$	$= (2, 7)-2^2(3, 6)+(2, 5)+3(4, 5)-(3, 4),$
(5.10)	$2^6.3^2(0, 1)^2(2, 3)$	$= 3(2, 7)-2.7(3, 6)+2.3(2, 5)+2^2.3(4, 5)-2^3.(3, 4)+(2, 3);$
(7.1)	$2^3.3.5(0, 3)^2$	$= (0, 7)-(0, 3),$
(7.2)	$2^3.3^2.7(0, 1)(0, 5)$	$= 2^2.5(0, 7)-2.3.7(1, 6)+3.7(0, 5)+(0, 1),$
(7.3)	$2^6.3^2.5(0, 1)^2(0, 3)$	$= 2.5(0, 7)-2.3.7(1, 6)+3.7(0, 5)+2^2.3^2(2, 5)$ $-2.3.5(1, 4)+2.3(1, 2)-(0, 1),$
(7.4)	$2^7.3^3(0, 1)^4$	$= 5(0, 7)-2.3.7(1, 6)+3.7(0, 5)+2^2.3^3(2, 5)-2.3^2.5(1, 4)$ $+3.5(0, 3)-2^3.3^2(3, 4)+2^3.3^2(2, 3)-2.3^2(1, 2)+(0, 1),$
(7.5)	$2^4.3.5(0, 3)(1, 4)$	$= (1, 8)-(1, 4),$
(7.6)	$2^3.3.7(0, 1)(1, 6)$	$= 5(1, 8)-2^2.3(2, 7)+7(1, 6),$

$$(7.7)\quad 2^3.3^2.7(1, 2)(0, 5) = 5(1, 8)-2.3(2, 7)+(1, 2),$$

$$(7.8)\quad 2^6.3^2.5(0, 1)^2(1, 4) = 5(1, 8)-2^3.3(2, 7)+2.7(1, 6)+2^3.3(3, 6)-2^3.3(2, 5)+5(1, 4),$$

$$\begin{aligned}(7.9)\quad 2^7.3^2.5(0, 1)(1, 2)(0, 3) = {} & 5(1, 8)-2.3^2(2, 7)+7(1, 6)+2^2.3(3, 6)\\ & -2.3(2, 5)-5(1, 4)+2.3(2, 3)-(1, 2),\end{aligned}$$

$$\begin{aligned}(7.10)\quad 2^9.3^3(0, 1)^3(1, 2) = {} & 5(1, 8)-2.3.7(2, 7)+3.7(1, 6)+2^2.3^3(3, 6)\\ & -2.3^2.5(2, 5)+3.5(1, 4)-2^3.3^2(2, 5)+3.5(1, 4)\\ & -2^3.3^2(4, 5)+2^3.3^2(3, 4)-2.3^2(2, 3)+(1, 2);\end{aligned}$$

$$(9.1)\quad 2^4.3^2.5.7(0, 3)(0, 5) = 11(0, 9)-3.7(0, 5)+2.5(0, 3),$$

$$(9.2)\quad 2^5.3.5(0, 1)(0, 7) = 11(0, 9)-2.3.5(1, 8)+2^2.5(0, 7)-(0, 1),$$

$$\begin{aligned}(9.3)\quad 2^8.3^2.5^2(0, 1)(0, 3)^2 = {} & 11(0, 9)-2.3.5(1, 8)+2^2.5(0, 7)-2.3.7(0, 5)\\ & +2^2.3.5(1, 4)-2^2.5(0, 3)+(0, 1),\end{aligned}$$

$$\begin{aligned}(9.4)\quad 2^6.3^3.7(0, 1)^2(0, 5) = {} & 11(0, 9)-2^2.3.5(1, 8)+2^3.5(0, 7)+2^3.3^2(2, 7)\\ & -2^2.3.7(1, 6)+3.7(0, 5)+2.5(0, 3)-2^2.3(1, 2)+2(0, 1),\end{aligned}$$

$$\begin{aligned}(9.5)\quad 2^{10}.3^3.5(0, 1)^3(0, 3) = {} & 11(0, 9)-2.3^2.5(1, 8)+2^2.3.5(0, 7)+2^3.3^3(2, 7)\\ & -2^2.3^2.7(1, 6)+2.3.7(0, 5)-2^4.3^2(3, 6)+2^3.3^3(2, 5)\\ & -2^2.5(0, 3)-2^3.3^2(2, 3)+2^2.3^2(1, 2)-3(0, 1),\end{aligned}$$

$$\begin{aligned}(9.6)\quad 2^{12}.3^4(0, 1)^5 = {} & 11(0, 9)-2.3.5^2(1, 8)+2^2.5^2(0, 7)+2^4.3^2.5(2, 7)\\ & -2^3.3.5.7(1, 6)+2.3.5.7(0, 5)-2^5.3^2.5(3, 6)\\ & +2^4.3^3.5(2, 5)-2^2.3^2.5^2(1, 4)+2^2.5^2(0, 3)\\ & +2^5.3^3(4, 5)-2^5.3^2.5(3, 4)+2^4.3^2.5(2, 3)\\ & -2^3.3.5(1, 2)+5(0, 1),\end{aligned}$$

$$\begin{aligned}(9.7)\quad 2^{12}.3^4.5(0, 1)^4(1, 2) = {} & 11(1, 10)-2.3.5^2(2, 9)+2^2.5^2(1, 8)+2^4.3^2.5(3, 8)\\ & -2^3.3.5.7(2, 7)+2.3.5.7(1, 6)-2^5.3^2.5(4, 7)\\ & +2^4.3^3.5(3, 6)-2^2.3^2.5^2(2, 5)+2^2.5^2(1, 4)\\ & +2^5.3^3(5, 6)-2^5.3^2.5(4, 5)+2^4.3^2.5(3, 4)\\ & -2^3.3.5(2, 3)+5(1, 2);\end{aligned}$$

$$(13.1)\quad 2^5.3^2.5.7(0, 5)(0, 7) = (0, 13)+2^2.5(0, 7)-3.7(0, 5),$$

$$(13.2)\quad 2^4.3.5.11(0, 3)(0, 9) = (0, 13)-11(0, 9)+2.5(0, 3),$$

$$(13.3)\quad 2^8.3^3.5^2.7(0, 3)^2(0, 5) = (0, 13)-2.11(0, 9)+2^2.5(0, 7)+3.7(0, 5)-2^2.5(0, 3),$$

$$(13.4)\quad 2^4.3^2.5.7.13(0, 1)(0, 11) = 691(0, 13)-2.3.5.7.13(1, 12)+2.3.5.7.13(0, 11)-691(0, 1).$$

Some of these results have been used by the author in a paper on the partition

function, recently published in the *Bulletin of the Calcutta Mathematical Society.* (Lahiri, 1946*b*).

4. The identities presented in Table B(1) have no doubt their own interest, but the main purpose of this paper being the exposition of a method of studying the congruence properties of $\sigma_k(n)$ (and also of $\tau(n)$), they are of greater interest because they provide us with a valuable tool for developing these properties. A glance at the Table will suggest that each of these identities gives rise to a congruence relation between $\sigma_k(n)$'s. By way of illustration we may state, (13 3) gives us on equating the coefficients of x^n on both sides of the identity

$$\sigma_{13}(n) \equiv 22\sigma_9(n) - 20\sigma_7(n) - 21\sigma_5(n) + 20\sigma_3(n) \pmod{2^9.3^3.5^2.7}.$$

But for the moment we shall refrain from putting down these congruences for we shall presently show that it is in general possible to derive more than one congruence relation from each of these identities. As a matter of fact we shall deduce nine congruences from (13.3), of which two involve $\sigma_k(n)$'s only, whereas the remaining involve $\tau(n)$ in addition.

It may be pointed out in this connection that the method adopted to obtain in general more than one relation from each of the identities in Table B(1) is essentially another method of studying the congruence properties of $\sigma_k(n)$, having interesting possibilities, but we shall here pursue the method only to that extent as required for the aforesaid purpose.

We shall now prove

Theorem (A). $$(u, v) = (u, s) + g_{v,s}J,$$

$$(u, v) = (u, s) + (r, v) - (r, s) + g_{u,r} \cdot g_{v,s}J,$$

where $g_{a,b} \mid n^a - n^b$, *whatever positive integral value be given to* n, *and* J *is a power series with integral coefficients.* (Throughout this paper we shall reserve the letter J to denote an integral power series, not necessarily the same in every case).

The theorem although important is quite simple to prove. It is in fact an immediate consequence of the algebraic identities

$$\text{(12)} \qquad \begin{cases} m^u n^v = m^u n^s + m^u(n^v - n^s), \\ m^u n^v = m^u n^s + m^r n^v - m^r n^s + (m^u - m^r)(n^v - n^s). \end{cases}$$

In what follows we shall require the values of $g_{a,b}$ for certain values of a and b. It is not difficult to show by using the Fermat-Euler theorem

$$n^{\phi(m)} \equiv 1 \pmod{m}, \text{ if } (m, n) = 1,$$

that

LEMMA 1.

$$
\begin{array}{lllll}
g_{3,1} = 2.3, & g_{5,1} = 2.3.5, & g_{7,1} = 2.3.7, & g_{9,1} = 2.3.5, & g_{11,1} = 2.3.11, \\
 & g_{5,3} = 2^3.3, & g_{7,3} = 2^3.3.5, & g_{9,3} = 2^3.3^2.7, & g_{11,3} = 2^3.3.5, \\
 & & g_{7,5} = 2^3.3, & g_{9,5} = 2^4.3.5, & g_{11,5} = 2^3.3^2.7, \\
 & & & g_{9,7} = 2^3.3, & g_{11,7} = 2^4.3.5, \\
 & & & & g_{11,9} = 2^3.3;
\end{array}
$$

$$
\begin{array}{llll}
g_{4,2} = 2^2.3, & g_{6,2} = 2^2.3.5, & g_{8,2} = 2^2.3^2.7, & g_{10,2} = 2^2.3.5, \\
 & g_{6,4} = 2^3.3, & g_{8,4} = 2^4.3.5, & g_{10,4} = 2^3.3^2.7, \\
 & & g_{8,6} = 2^3.3, & g_{10,6} = 2^4.3.5, \\
 & & & g_{10,8} = 2^3.3.
\end{array}
$$

6. All congruences involving $\sigma_k(n)$ which are derivable from the identities in Table B(1) by making use of Theorem A are given in the next Table C(1). The method of derivation is best illustrated by the following examples.

Example 1. We have the formula B(1) (3.6),

$$2^3.3(0, 1)(3, 4) = 2(3, 6)-3(4, 5)+(3, 4). \tag{13}$$

Also by Theorem A

$$(3, 4) = (3, 2)+(1, 4)-(1, 2)+g_{3,1}.g_{4,2}J,$$

or,

$$(3, 4) = (2, 3)+(1, 4)-(1, 2)+2^3.3^2J.$$

Thus using identities B(1) (3.2), (3.4), (5.3) we get

$$
\begin{aligned}
(14)\quad 2^3.3(0, 1)(3, 4) &= 2^3.3(0, 1)(2, 3)+\tfrac{1}{5}\times 2^3.3.5(0, 1)(1, 4)-2^3.3(0, 1)(1, 2)+2^6.3^3J, \\
&= \{3(2, 5)-4(3, 4)+(2, 3)\}+\tfrac{1}{5}\{7(1, 6)-12(2, 5)+5(1, 4)\} \\
&\qquad -\{5(1, 4)-6(2, 3)+(1, 2)\}+2^6.3^3J.
\end{aligned}
$$

Now (13) and (14) lead to

$$7(1, 6) = \{10(3, 6)-3(2, 5)+20(1, 4)\}-5\{3(4, 5)-5(3, 4)+7(2, 3)-(1, 2)\}+2^6.3^3.5J.$$

Finally on equating the coefficients of x^n on both sides we get

$$7n\sigma_5(n) \equiv (10n^3-3n^2+20n)\sigma_3(n)-5(3n^4-5n^3+7n^2-n)\sigma(n) \pmod{2^6.3^3.5}.$$

Example 2. We start now from the identity B(1) (5.8)

$$2^4.3.5(2, 3)(0, 3) = 3(2, 7)-2(3, 6)-(2, 3). \tag{15}$$

Also by Theorem A

$$(2, 3) = (2, 1)+g_{3,1}J,$$

$$(0, 3) = (0, 1)+g_{3,1}J;$$

and these give us

$$(2, 3)(0, 3) = (0, 1)(2, 3)+(1, 2)(0, 3)-(0, 1)(1, 2)+g_{3,1}^2J.$$

Therefore on using B(1) (3.2), (3.4), (5.4) we get

$$(16)\quad 2^4.3.5(2,\ 3)(0,\ 3) = 10\times 2^3.3(0,\ 1)(2,\ 3) + 2^4.3.5(1,\ 2)(0,\ 3) - 10\times 2^3.3(0,\ 1)(1,\ 2) + 2^6.3^3.5J$$

$$= 10\{3(2,\ 5) - 4(3,\ 4) + (2,\ 3)\} + \{7(1,\ 6) - 6(2,\ 5) - (1,\ 2)\} - 10\{5(1,\ 4) - 6(2,\ 3) + (1,\ 2)\} + 2^6.3^3.5J.$$

Now (15) and (16) give us

$$3(2,\ 7) - 7(1,\ 6) = \{2(3,\ 6) + 24(2,\ 5) - 50(1,\ 4)\} - \{40(3,\ 4) - 71(2,\ 3) + 11(1,\ 2)\} + 2^6.3^3.5J.$$

Equating the coefficients of x^n on both sides we get

$$(3n^2 - 7n)\sigma_5(n) \equiv 2(n^3 + 12n^2 - 25n)\sigma_3(n) - (40n^3 - 71n^2 + 11n)\sigma(n) \pmod{2^6.3^3.5}.$$

In Table C(1) the figures within square brackets indicate the identity in Table B(1) from which the congruence relation is derived by the appropriate use of Theorem A.

TABLE C(1)

(3.1)	$5\sigma_3(n)$	$\equiv (6n-1)\sigma(n)$	(mod $2^2.3$),	[3.1]
(3.2)	$n^2\sigma_3(n)$	$\equiv n^3\sigma(n)$	(mod $2^2.3$),	[3.3]
(3.3)	$n^3\sigma_3(n)$	$\equiv \frac{1}{2}(3n^4 - n^3)\sigma(n)$	(mod $2^2.3$),	[3.6]
(3.4)	$5n\sigma_3(n)$	$\equiv (6n^2 - n)\sigma(n)$	(mod $2^3.3$),	[3.2]
(3.5)	$3n^2\sigma_3(n)$	$\equiv (4n^3 - n^2)\sigma(n)$	(mod $2^3.3$),	[3.4]
(3.6)	$n^3\sigma_3(n)$	$\equiv n^4\sigma(n)$	(mod $2^3.3$),	[3.5]
(3.7)	$\frac{1}{2}(3n^2 - 5n)\sigma_3(n)$	$\equiv \frac{1}{2}(4n^3 - 7n^2 + n)\sigma(n)$	(mod $2^3.3^2$),	[3.4]
(3.8)	$(n^3 - 2n^2)\sigma_3(n)$	$\equiv (n^4 - 2n^3)\sigma(n)$	(mod $2^4.3^2$),	[3.5]
(3.9)	$(2n^3 - 3n^2)\sigma_3(n)$	$\equiv (3n^4 - 5n^3 + n^2)\sigma(n)$	(mod $2^5.3^2$) ;	[3.6]
(5.1)	$n^2\sigma_5(n)$	$\equiv (2n^3 - n^2)\sigma_3(n)$	(mod $2^3.3$),	[5.7]
(5.2)	$7\sigma_5(n)$	$\equiv 10(3n-1)\sigma_3(n) - (24n^2 - 12n + 1)\sigma(n)$	(mod $2^6.3$),	[5.2]
(5.3)	$n^3\sigma_5(n)$	$\equiv (4n^3 - n^2)\sigma_3(n) - (3n^4 - n^3)\sigma(n)$	(mod $2^5.3^2$),	[5.9]
(5.4)	$7n\sigma_5(n)$	$\equiv 10(3n^2 - n)\sigma_3(n) - (24n^3 - 12n^2 + n)\sigma(n)$	(mod $2^6.3^2$),	[5.5]
(5.5)	$n^2\sigma_5(n)$	$\equiv 2(n^3 + n^2)\sigma_3(n) - (4n^3 - n^2)\sigma(n)$	(mod $2^6.3^2$),	[5.7]
(5.6)	$3n^2\sigma_5(n)$	$\equiv 2(7n^3 - 3n^2)\sigma_3(n) - (12n^4 - 8n^3 + n^2)\sigma(n)$	(mod $2^6.3^2$),	[5.10]
(5.7)	$(3n^2 - 7n)\sigma_5(n)$	$\equiv 2(7n^3 - 18n^2 + 5n)\sigma_3(n) - (12n^4 - 32n^3 + 13n^2 - n)\sigma(n)$	(mod $2^7.3^3$),	[5.10]
(5.8)	$n^3\sigma_5(n)$	$\equiv n^3\sigma_3(n)$	(mod $2^2.3.5$),	[5.6]
(5.9)	$7n\sigma_5(n)$	$\equiv (12n^2 - 5n)\sigma_3(n)$	(mod $2^3.3.5$),	[5.3]
(5.10)	$21\sigma_5(n)$	$\equiv 10(3n-1)\sigma_3(n) + \sigma(n)$	(mod $2^4.3.5$),	[5.1]
(5.11)	$7n\sigma_5(n)$	$\equiv 6n^2\sigma_3(n) + n\sigma(n)$	(mod $2^4.3.5$),	[5.4]

$$(5.12)\quad 3n^2\sigma_5(n) \equiv 2n^3\sigma_3(n)+n^2\sigma(n) \pmod{2^4.3.5}, \quad [5.8]$$

$$(5.13)\quad 7\sigma_5(n) \equiv 10(n+3)\sigma_3(n)-(40n-7)\sigma(n) \pmod{2^5.3.5}, \quad [5.1]$$

$$(5.14)\quad 7n\sigma_5(n) \equiv (10n^3+12n^2-5n)\sigma_3(n)-5(3n^4-n^3)\sigma(n) \pmod{2^4.3^2.5}, \quad [5.6]$$

$$(5.15)\quad n^2\sigma_5(n) \equiv (n^3+5n^2)\sigma_3(n)-5n^3\sigma(n) \pmod{2^4.3^2.5}, \quad [5.6]$$

$$(5.16)\quad n^2\sigma_5(n) \equiv (2n^3-n^2+5n)\sigma_3(n)-(6n^2-n)\sigma(n) \pmod{2^4.3^2.5}, \quad [5.7]$$

$$(5.17)\quad 7n\sigma_5(n) \equiv 4(3n^2+5n)\sigma_3(n)-5(6n^2-n)\sigma(n) \pmod{2^5.3^2.5}, \quad [5.3]$$

$$(5.18)\quad 7n\sigma_5(n) \equiv 2(3n^2+25n)\sigma_3(n)-(60n^2-11n)\sigma(n) \pmod{2^5.3^2.5}, \quad [5.4]$$

$$(5.19)\quad 3n^2\sigma_5(n) \equiv 2(n^3+15n^2)\sigma_3(n)-(40n^3-11n^2)\sigma(n) \pmod{2^5.3^2.5}, \quad [5.8]$$

$$(5.20)\quad (3n^2-7n)\sigma_5(n) \equiv 2(n^3-3n^2)\sigma_3(n)+(n^2-n)\sigma(n) \pmod{2^5.3^2.5}, \quad [5.8]$$

$$(5.21)\quad 7n\sigma_5(n) \equiv (10n^3-3n^2+20n)\sigma_3(n) - 5(3n^4-5n^3+7n^2-n)\sigma(n) \pmod{2^6.3^3.5}, \quad [3.6]$$

$$(5.22)\quad (3n^2-7n)\sigma_5(n) \equiv 2(n^3+12n^2-25n)\sigma_3(n) - (40n^3-71n^2+11n)\sigma(n) \pmod{2^6.3^3.5}; \quad [5.8]$$

$$(7.1)\quad 5\sigma_7(n) \equiv 21(2n-1)\sigma_5(n)-3(36n^2-30n+5)\sigma_3(n) + (72n^3-72n^2+18n-1)\sigma(n) \pmod{2^7.3^3}, \quad [7.4]$$

$$(7.2)\quad 5n\sigma_7(n) \equiv 21(2n^2-n)\sigma_5(n)-3(36n^3-30n^2+5n)\sigma_3(n) + (72n^4-72n^3+18n^2-n)\sigma(n) \pmod{2^9.3^3}, \quad [7.10]$$

$$(7.3)\quad \sigma_7(n) \equiv \sigma_3(n) \pmod{2^3.3.5}, \quad [7.1]$$

$$(7.4)\quad n\sigma_7(n) \equiv n\sigma_3(n) \pmod{2^4.3.5}, \quad [7.5]$$

$$(7.5)\quad 2\sigma_7(n) \equiv 21\sigma_5(n)-6(5n-2)\sigma_3(n)-\sigma(n) \pmod{2^5.3^2.5}, \quad [7.1]$$

$$(7.6)\quad n\sigma_7(n) \equiv 14n\sigma_5(n)-(24n^2-11n)\sigma_3(n) \pmod{2^5.3^2.5}, \quad [7.5]$$

$$(7.7)\quad 10\sigma_7(n) \equiv 21(2n-1)\sigma_5(n)-6(6n^2-5n)\sigma_3(n) - (6n-1)\sigma(n) \pmod{2^6.3^2.5}, \quad [7.3]$$

$$(7.8)\quad n\sigma_7(n) \equiv 7n\sigma_5(n)-(6n^2-n)\sigma_3(n)-n\sigma(n) \pmod{2^6.3^2.5}, \quad [7.5]$$

$$(7.9)\quad 5n\sigma_7(n) \equiv 2(12n^2-7n)\sigma_5(n)-(24n^3-24n^2+5n)\sigma_3(n) \pmod{2^6.3^2.5}, \quad [7.8]$$

$$(7.10)\quad 5n\sigma_7(n) \equiv (18n^2-7n)\sigma_5(n)-(12n^3-6n^2-5n)\sigma_3(n) - (6n^2-n)\sigma(n) \pmod{2^7.3^2.5}, \quad [7.9]$$

$$(7.11)\quad \sigma_7(n) \equiv 21\sigma_5(n)-3(10n+13)\sigma_3(n)+(60n-11)\sigma(n) \pmod{2^5.3^3.5}, \quad [7.1]$$

$$(7.12)\quad 5\sigma_7(n) \equiv 21(n+2)\sigma_5(n)-3(6n^2+70n-25)\sigma_3(n) + (180n^2-93n+8)\sigma(n) \pmod{2^6.3^3.5}, \quad [7.3]$$

$$(7.13)\quad n\sigma_7(n) \equiv 21n\sigma_5(n)-3(10n^2+13n)\sigma_3(n) + (60n^2-11n)\sigma(n) \pmod{2^7.3^3.5}, \quad [7.5]$$

$$(7.14)\quad 5n\sigma_7(n) \equiv 3(8n^2+7n)\sigma_5(n)-3(8n^3+42n^2-15n)\sigma_3(n) + (24n^3-12n^2+n)\sigma(n) \pmod{2^8.3^3.5}, \quad [7.8]$$

$$\begin{aligned}
(7.15)\quad 5n\sigma_7(n) &\equiv 9(2n^2+7n)\sigma_5(n)-3(4n^3+98n^2-35n)\sigma_3(n) \\
&\quad +(240n^3-126n^2+11n)\sigma(n) && (\text{mod } 2^5.3^3.5), && [7.9] \\
(7.16)\quad 5n\sigma_7(n) &\equiv (12n^2-7n)\sigma_5(n) && (\text{mod } 2^3.3.7), && [7.6] \\
(7.17)\quad 20\sigma_7(n) &\equiv 21(2n-1)\sigma_5(n)-\sigma(n) && (\text{mod } 2^3.3^2.7), && [7.2] \\
(7.18)\quad 5n\sigma_7(n) &\equiv 6n^2\sigma_5(n)-n\sigma(n) && (\text{mod } 2^3.3^2.7), && [7.7] \\
(7.19)\quad 5n\sigma_7(n) &\equiv (12n^2-7n)\sigma_5(n)+35n\sigma_3(n)-7(6n^2-n)\sigma(n) && (\text{mod } 2^6.3^2.5.7), && [7.6] \\
(7.20)\quad 25n\sigma_7(n) &\equiv 2(30n^2+7n)\sigma_5(n)-7(12n^2-5n)\sigma_3(n) && (\text{mod } 2^6.3^2\,5.7), && [7.6] \\
(7.21)\quad 20\sigma_7(n) &\equiv 21(2n-1)\sigma_5(n)+210\sigma_3(n)-(252n-41)\sigma(n) && (\text{mod } 2^4.3^3.5.7), && [7.2] \\
(7.22)\quad 5n\sigma_7(n) &\equiv 6n^2\sigma_5(n)+105n\sigma_3(n)-2(63n^2-10n)\sigma(n) && (\text{mod } 2^4.3^3.5.7), && [7.7] \\
(7.23)\quad 200\sigma_7(n) &\equiv 21(20n+11)\sigma_5(n)-210(3n-1)\sigma_3(n)-31\sigma(n) && (\text{mod } 2^7.3^3.5.7), && [7.2] \\
(7.24)\quad 50n\sigma_7(n) &\equiv 3(20n^2+49n)\sigma_5(n)-126n^2\sigma_3(n)-31n\sigma(n) && (\text{mod } 2^7.3^3.5.7); && [7.7]
\end{aligned}$$

$$\begin{aligned}
(9.1)\quad 11\sigma_9(n) &\equiv 50(3n-2)\sigma_7(n)-30(24n^2-28n+7)\sigma_5(n) \\
&\quad +20(72n^3-108n^2+45n-5)\sigma_3(n) \\
&\quad -(864n^4-1440n^3+720n^2-120n+5)\sigma(n) && (\text{mod } 2^{12}.3^4), && [9.6] \\
(9.2)\quad 11\sigma_9(n) &\equiv 10(3n-2)\sigma_7(n)+\sigma(n) && (\text{mod } 2^5.3.5), && [9.2] \\
(9.3)\quad 11\sigma_9(n) &\equiv 30(3n-2)\sigma_7(n)-6(36n^2-42n+7)\sigma_5(n) \\
&\quad +4(36n^3-54n^2+5)\sigma_3(n) \\
&\quad +(72n^2-36n+3)\sigma(n) && (\text{mod } 2^{10}.3^3.5), && [9.5] \\
(9.4)\quad 11\sigma_9(n) &\equiv 10(9n+14)\sigma_7(n)-6(36n^2+238n \\
&\quad -183)\sigma_5(n)+4(36n^3+1026n^2-900n \\
&\quad +155)\sigma_3(n)-(2880n^3-2952n^2 \\
&\quad +756n-43)\sigma(n) && (\text{mod } 2^{11}.3^4.5), && [9.5] \\
(9.5)\quad 11n\sigma_9(n) &\equiv 50(3n^2-2n)\sigma_7(n)-30(24n^3-28n^2 \\
&\quad +7n)\sigma_5(n)+20(72n^4-108n^3+45n^2 \\
&\quad -5n)\sigma_3(n)-(864n^5-1440n^4 \\
&\quad +720n^3-120n^2+5n)\sigma(n) && (\text{mod } 2^{12}.3^4.5), && [9.7] \\
(9.6)\quad 11\sigma_9(n) &\equiv 10(3n-2)\sigma_7(n)+42\sigma_5(n) \\
&\quad -20(3n-1)\sigma_3(n)-\sigma(n) && (\text{mod } 2^8.3^2.5^2), && [9.2, 9.3] \\
(9.7)\quad 11\sigma_9(n) &\equiv 30(n+6)\sigma_7(n)-42(20n-11)\sigma_5(n) \\
&\quad +20(36n^2-33n+1)\sigma_3(n)+3(40n-7)\sigma(n) && (\text{mod } 2^9.3^3.5^2), && [9.3] \\
(9.8)\quad 11\sigma_9(n) &\equiv 10(3n+38)\sigma_7(n)-6(280n+203)\sigma_5(n) \\
&\quad +20(72n^2+387n-149)\sigma_3(n) \\
&\quad -(7200n^2-3840n+341)\sigma(n) && (\text{mod } 2^{10}.3^4.5^2), && [9.3]
\end{aligned}$$

(9.9)	$11\sigma_9(n)$	$\equiv 20(3n-2)\sigma_7(n)-3(24n^2-28n+7)\sigma_5(n)$ $-10\sigma_3(n)+2(6n-1)\sigma(n)$	(mod $2^6.3^3.7$),	[9.4]
(9.10)	$11\sigma_9(n)$	$\equiv 21\sigma_5(n)-10\sigma_3(n)$	(mod $2^4.3^2.5.7$),	[9.1]
(9.11)	$11\sigma_9(n)$	$\equiv 10(3n-2)\sigma_7(n)+200\sigma_3(n)-(240n-41)\sigma(n)$	(mod $2^6.3^2.5.7$),	[9.2]
(9.12)	$11\sigma_9(n)$	$\equiv 200\sigma_7(n)-(420n-231)\sigma_5(n)-10\sigma_3(n)+10\sigma(n)$	(mod $2^5.3^3.5.7$),	[9.1]
(9.13)	$11\sigma_9(n)$	$\equiv 42\sigma_7(n)+21\sigma_5(n)-52\sigma_3(n)$	(mod $2^7.3^3.5.7$),	[9.1]
(9.14)	$231\sigma_9(n)$	$\equiv 10(63n-2)\sigma_7(n)-420(2n-1)\sigma_5(n)+41\sigma(n)$	(mod $2^8.3^3.5.7$),	[9.2]
(9.15)	$11\sigma_9(n)$	$\equiv 20(3n-2)\sigma_7(n)-12(6n^2-7n-35)\sigma_5(n)$ $-10(189n-62)\sigma_3(n)$ $+(1512n^2-744n+61)\sigma(n)$	(mod $2^7.3^4.5.7$),	[9.4]
(9.16)	$11\sigma_9(n)$	$\equiv 242\sigma_7(n)-210(2n+1)\sigma_5(n)$ $+2(315n-131)\sigma_3(n)+31\sigma(n)$	(mod $2^9.3^4.5.7$),	[9.1]
(9.17)	$55\sigma_9(n)$	$\equiv 10(30n+1)\sigma_7(n)-6(60n^2+77n-56)\sigma_5(n)$ $+2(378n^2-315n-25)\sigma_3(n)+31(6n-1)\sigma(n)$	(mod $2^9.3^4.5.7$),	[9.4]
(9.18)	$11\sigma_9(n)$	$\equiv 462\sigma_7(n)+21\sigma_5(n)-52\sigma_3(n)$	(mod $2^5.3^3.5^2.7$),	[9.1]
(9.19)	$11\sigma_9(n)$	$\equiv 200\sigma_7(n)-84(5n-8)\sigma_5(n)$ $-10(63n+190)\sigma_3(n)+(2520n-431)\sigma(n)$	(mod $2^6.3^4.5^2.7$);	[9.1]
(13.1)	$\sigma_{13}(n)$	$\equiv -20\sigma_7(n)+21\sigma_5(n)$	(mod $2^3.3^2.5.7$),	[13.1]
(13.2)	$\sigma_{13}(n)$	$\equiv 231\sigma_9(n)-10(63n-40)\sigma_7(n)$ $+21\sigma_5(n)-21\sigma(n)$	(mod $2^6.3^3.5^2.7$),	[13.1]
(13.3)	$\sigma_{13}(n)$	$\equiv 22\sigma_9(n)-20\sigma_7(n)-21\sigma_5(n)+20\sigma_3(n)$	(mod $2^9.3^3.5^2.7$),	[13.1, 13.3]
(13.4)	$\sigma_{13}(n)$	$\equiv 253\sigma_9(n)-10(63n-40)\sigma_7(n)$ $-903\sigma_5(n)+20(63n-20)\sigma_3(n)+21\sigma(n)$	(mod $2^9.3^4.5^3.7$),	[13.1, 13.3]
(13.5)	$\sigma_{13}(n)$	$\equiv 380\sigma_7(n)-21(40n-21)\sigma_5(n)+20\sigma(n)$	(mod $2^6.3^3.5.7^2$),	[13.1]
(13.6)	$\sigma_{13}(n)$	$\equiv 231\sigma_9(n)-10(63n-80)\sigma_7(n)-21(40n$ $-21)\sigma_5(n)-4200\sigma_3(n)+(5040n-841)\sigma(n)$	(mod $2^7.3^4.5^2.7^2$),	[13.1]
(13.7)	$\sigma_{13}(n)$	$\equiv 11\sigma_9(n)-10\sigma_3(n)$	(mod $2^4.3.5.11$),	[13.2]
(13.8)	$\sigma_{13}(n)$	$\equiv 11\sigma_9(n)+231\sigma_5(n)$ $-10(33n-10)\sigma_3(n)-11\sigma(n)$	(mod $2^5.3^2.5^2.11$),	[13.2]
(13.9)	$\sigma_{13}(n)$	$\equiv 11\sigma_9(n)+22\sigma_7(n)-32\sigma_3(n)$	(mod $2^7.3^3.5.7.11$),	[13.2]
(13.10)	$21\sigma_{13}(n)$	$\equiv 352\sigma_9(n)-231\sigma_5(n)-100\sigma_7(n)$	(mod $2^5.3^3.5^2.7.11$),	[13.2]
(13.11)	$691\sigma_{13}(n)$	$\equiv 2730(n-1)\sigma_{11}(n)+691\sigma(n)$	(mod $2^4.3^2.5.7.13$),	[13.4]
(13.12)	$691\sigma_{13}(n)$	$\equiv 2730(n-1)\sigma_{11}(n)+5733\sigma_5(n)$ $-2730(3n-1)\sigma_3(n)+418\sigma(n)$	(mod $2^7.3^3.5^2.7.13$),	[13.4]

$$(13.13)\quad 1382\sigma_{13}(n) \equiv 5460(n-1)\sigma_{11}(n)+3003\sigma_{9}(n) \\ -2730(3n-2)\sigma_{7}(n)+1109\sigma(n) \pmod{2^{9}.3^{3}.5^{2}.7.13}, \qquad [13.4]$$

$$(13.14)\quad 691\sigma_{13}(n) \equiv 2730(n-1)\sigma_{11}(n)+2600\sigma_{7}(n) \\ -2730(2n-1)\sigma_{5}(n)+821\sigma(n) \pmod{2^{7}.3^{4}.5.7^{2}.13}, \qquad [13.4]$$

$$(13.15)\quad 691\sigma_{13}(n) \equiv 2730(n-1)\sigma_{11}(n)+27300\sigma_{3}(n) \\ -(32760n-6151)\sigma(n) \pmod{2^{5}.3^{3}.5.7.11.13}. \qquad [13.4]$$

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CORRECTIONS TO MY PAPER ON "PARALLELISM IN RIEMANNIAN SPACE"

By

R. N. Sen

This Bulletin, Vol. 36, No. 3 (1944), page 102

for	*read*
$\{T^i_{ij}\}_k - \{T^i_{ik}\}_j$	$\{T^i_{ik}\}_j - \{T^i_{ij}\}_k$
consideration	consideration on the supposition that one of the parallel displacements, say (1.2), is that of Levi-Civita
$\sum\limits_i \lambda_i \mu^i$	$\cos\theta = \sum\limits_i \lambda_i \mu^i$
the covariant . . . vectors	the contravariant and covariant vectors

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ERRATA

A NOTE ON BIANCHI CONGRUENCE

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The name of the author will be Ratan Shanker Mishra in place of Rajen Shanker Mishra.